$j \cdots jj$-planes

Algebraic Combinatorics Seminar, Ft. Collins. September 29, 2006.

Abstract

A new family of translation planes is constructed and studied, it generalizes one found by Johnson, Pomareda and Wilke (J. Combin. Theory Ser. A 56 (1991), 272-284). Two more families of planes may be constructed from the one found by replacing some of the lines of these planes with different point-sets that play the role of lines. These processes will be explained in detail.

Finally, a subfamily of the planes constructed yields new partitions of Segre varieties by Veroneseans (flat flocks).

Spreads

Definition: Let V be a $2n$-dimensional vector space over a field K. A spread S of V is a set of n-dimensional subspaces of V that intersect trivially and that partition the space.

The elements of S are called components of the spread S.

The direct sum of any two components of S is equal to V.

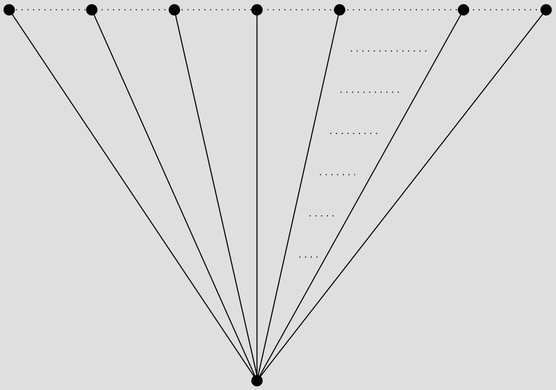

Fig. 2: Spread.

Spreads and translation planes

Let V_{2n} be a $2n$-dimensional vector space over a field K, and let S be a spread of V. We define $\Pi = (P, L, \Im)$ by:

i. The elements of P are the points (vectors) of V_{2n}

ii. The elements of L are the components of S and all its (additive) cosets.

iii. The incidence is given by the natural set theoretic inclusion.

Theorem: Π is an Affine Plane. Moreover, if $K = \mathbb{F}_q$ then, the order of Π is q^n.

Theorem: (André) Every translation plane can be represented by using a spread.

Notation for lines

As $V_{2n} \cong K^n \oplus K^n$, we can think points of V_{2n} to look like (x, y), with $x, y \in K^n$.

For a fixed matrix $M \in M_n(K)$, call $(y = xM)$ to the n-dimensional subspace

$$\{(x, y) \in V;\ y = xM\}\,.$$

Note that any n-dimensional subspace of V that is disjoint from $(x = 0)$ can be represented as $(y = xM)$, for some suitable $n \times n$ matrix M.

To the infinity, and beyond...

When studying collineations of translation planes, it is very helpful to consider an extra line, that is **not** in the affine plane.

Definition Let Π be a translation plane with spread S, then

$$\ell_\infty = \{\ (M);\ M \in S\} \cup \{(\infty)\}$$

is called the line at infinity of Π.

One may think ℓ_∞ as the set of all "slopes" of lines in Π. In this case, the point (∞) represents the slope of the line $(x = 0)$.

$\Pi \cup \ell_\infty$ is a projective plane of order q^n.

Example of a spread

Let $p(x) = x^3 - ax^2 - bx - c$ be irreducible in $\mathbb{F}_q[x]$.

It is not hard to see that the set E of all matrices of the form

$$M_{r,s,t} = \begin{bmatrix} r & s & t \\ ct & r + bt & s + at \\ c(s + at) & ct + b(s + at) & r + bt + a(s + at) \end{bmatrix},$$

where $r, s, t \in \mathbb{F}_q$, is a field of order q^3 contained in $GL(3, q) \cup \{0\}$.

Define $S = \{(x = 0)\} \cup \{(y = xM)\ ;\ M \in E\}$.

The fact that E is a field implies that S is a spread. Its associated translation plane has order q^3.

A more general field of matrices

Consider $K = \{\alpha Id \in M_n(q);\ \alpha \in \mathbb{F}_q\}$, and the polynomial $p(x) = x^n - a_{n-1}x^{n-1} - \cdots - a_1 x - a_0$, irreducible over $\mathbb{F}_q[x]$.

Then

$$\theta := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \end{bmatrix}$$

is the companion matrix of $p(x)$, thus $p(\theta) = 0$. Since $F = K(\theta)$ is a field isomorphic to $\mathbb{F}_{q^n}$, then

$$S = \{(x = 0)\} \cup \{(y = xM)\ ;\ M \in F\}$$

is a spread of “order” q^n.

$jj\cdots j$-planes

Let $j_2, j_3, ..., j_n \in \{0, 1, ..., q-2\}$ and let $F = K(\theta)$. Define:

$$G = \left\{ \begin{bmatrix} \Delta_M^{-1} & 0 \\ 0 & M \end{bmatrix} ; \; M \in F^* \right\} \subset M_{2n}(q)$$

where $\Delta_M = diag(1, \partial^{j_2}, \partial^{j_3}, ..., \partial^{j_n})$ and $\partial = \det(M)$.

It is easy to see that G is a cyclic group of order $q^n - 1$.

In case $S = \{(x = 0), (y = 0)\} \cup O_G(y = x)$ is a spread, we say that its associated translation plane is a $j_2 j_3 \cdots j_n$-plane, or simply a $jj\cdots j$-plane.

A few basic results

1. A $j_2 \cdots j_n$-plane is Desarguesian, if and only if, $j_2 = \cdots j_n = 0$.

2. It is OK to consider $j_1 = 0$.

3. $S = \{(y = 0)\} \cup O_G(y = x)$ is a spread, if and only if,

$$\det(\Delta_M M - Id) \neq 0 \text{ for every } M \neq Id \text{ in } F.$$

Actually, it is enough to check this for only half of F, as

$$\det(\Delta_M M - Id) \neq 0 \iff \det(\Delta_{M^{-1}} M^{-1} - Id) \neq 0.$$

4. Let Π be a $j_2 j_3 \ldots j_n - plane$ of order q^n, then

$$\gcd(nj_i + 1, q - 1) = 1 \text{ for every } i > 1.$$

Existence

Using the previous results, some educated guessing, and Maple ©, we were able to find all $jj\cdots j$-planes of orders 4^3, 7^3, 3^4, 4^4 and 5^4. Also, using the same techniques we could show that there are no $jj\cdots j$-planes of orders neither 3^n (for n odd) nor 5^3 but the Desarguesians.

An infinite family of André $jj\cdots j$-planes of order q^n was found for when $q-1=nk$ and $\gcd(n,k)=1$. This class of planes is constructed over a field of matrices F is given by a polynomial of the form $p(x)=x^n-c$, where $< c >= \mathbb{F}_q^*$.

Collineations

Definition A collineation of an affine plane Π is an injective map that preserves the incidence relation.

Definition An isomorphism between two planes Π_1 and Π_2 is a bijection from the points of Π_1 onto the points of Π_2 that preserves the incidence relation.

Definition If a collineation Ψ of Π fixes a line ℓ pointwise and all the lines through a point P setwise, then Ψ is called a perspectivity.

1) If $P \in \ell$, then Ψ is called and elation.

2) If $P \notin \ell$, then Ψ is called a homology.

In either case, P is called the center of Ψ and ℓ is called the axis of Ψ. When Ψ is an affine homology, the component that contains the center is called the coaxis of Ψ.

Full collineation group of a $jj\cdots j$-plane

Let $\mathbb{T}$, be the translations group. $\mathbb{T}$ is a collineation group of any translation plane. The full collineation group of a translation plane is the semi-direct product of $\mathbb{T}$ and a group $C \leq \Gamma L(2n, q)$, which is called the translation complement.

Theorem: The translation complement of a non-André $jj\cdots j$-plane Π fixes the lines $(x = 0)$ and $(y = 0)$. Moreover, the linear part of it is isomorphic to the direct product of G and the kernel homologies.

New?? Isomorphism classes??

Theorem: A non-André $jj\cdots j$-plane is new.

The proof is technical, but the idea is very simple. Since we know the full collineation group of a $j\cdots jj$-plane, then one just needs to show that all known translation planes do not admit that group. For example,

1. If Π were a nearfield plane, then because its order is not a prime number or the square of a prime number, it would be André.

2. If Π were generalized André, since the plane has a homology group of size $(q^n-1)/(q-1)$, then the plane would be André.

3. If Π is a semifield plane, then the homology group of order $(q^n-1)/(q-1)$ that Π admits forces the nuclei to have order at least $(q^n-1)/(q-1)+1$. Then the semifield becomes a skewfield and Π is Desarguesian.

4. The orbits under the full collineation group discard flag-transitive planes, triangle transitive (does not act transitively on $(x=0)$), $SL(2,q)$-planes, cubic Figueroa (no collineation fixing the lines of a subplane of order q), and generalized André (does not fix the lines in ℓ_∞ of a Baer subplane).

Another series of technical results like the previous ones were necessary to determine the isomorphism classes of $jj\cdots j$-planes. I have found several isomorphisms that work independently of the order of the planes. Moreover, for the "sporadic" planes found by computer, and for other smaller classes of planes.

When $p(x) = x^n - c$

When the field of matrices F is given by a polynomial of the form $p(x) = x^n - c$, then several isomorphism between $jj \cdots j$-planes can be constructed that are independent of q and n. Moreover, more restrictions may be imposed on the j's, which is important theoretically and for computing. One of the nice results these new restrictions yielded is that the spreadset of a $jj \cdot j$-plane cannot be contained in $SL(n, q)$.

I believe the study of these planes is important, they seem to carry all the information necessary to investigate all $jj \cdots j$-planes.

Replacement and derivation on planes

The processes of replacement and derivation of planes consist in removing a certain set of lines of the plane and replace them by other suitable set of subspaces. A new translation plane is obtained.

Not every plane is replaceable or derivable.

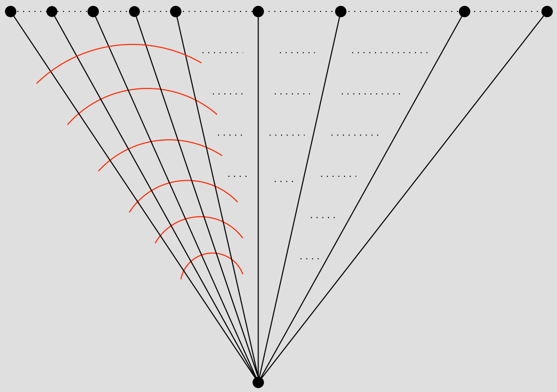

Fig. 2: Derivation in a plane.

Replaceable nets

The sets of lines we will replace to create new planes are called replaceable nets or derivable nets.

Definition Let N be a subset of lines of a translation plane π such that there exists a set N' of "disjoint" n-dimensional subspaces over $GF(q)$ that cover the same points N covers, and with the property that any two points joined by a line of N are also joined by a subspace of N'.

N is called a replaceable net and N' is said to be its replacement. Moreover, $\Pi = (\pi \setminus N) \cup N'$ is a translation plane.

Towards derivation

In order to have a derivable net in a translation plane π, we need π to be of square order. Let us say that the order of π is $q^n = h^2$.

Consider π_0, an affine subplane of π of order h. We can see that the number of points in π_0 is $h^2 = q^n$, exactly the same number of points as an n-dimensional vector space over $GF(q)$.... same as the number of points of a component of π!!!!. Also, note that the number of components in the spread of π_0 is h.

So, the plan is to find a set N of h components of π that can be covered by a set D of “disjoint” subplanes, which will eventually replace the lines of N to create a new plane.

Derivable nets

In order to get a plane after replacing lines by subplanes, we need that if two points are collinear in N, then there must be a subplane in D that also contains both points.

Definition Whenever a set N of lines and a set D of subplanes satisfy the situation previously discussed, it will be called a derivable net. The set D is its derived net, and $\Pi = (\pi \setminus N) \cup D$ will be said to be the derived plane of π.

Note that the lines of Π are not necessarily subspaces over $GF(q)$.

How to obtain replaceable and derivable nets in $jj...j$-planes???.

Homologies

Homology groups are probably the most important collineation groups of a translation plane. Also, sometimes they may induce replaceable and/or derivable nets.

The subgroup of G given by:

$$\Gamma = \left\{ \begin{bmatrix} \Delta_M^{-1} & 0 \\ 0 & M \end{bmatrix} \in G;\ det(M) = 1 \right\}$$

is a cyclic $((y=0),(\infty))$-homology group of order $(q^n-1)/(q-1)$.

Homologies induce replaceable nets

The orbits of Γ define $q-1$ (disjoint) replaceable nets;

$$N_v = \{(y = x\Delta M);\ \det(M) = v\}$$

for $v \in GF(q)^*$.

Each N_v has $n-1$ distinct replacements, they are:

$$N_v^{(t)} = \left\{(y = x^{q^t}\Delta M);\ \det(M) = v\right\}$$

for $t = 1, 2, ..., n-1$.

Homologies induce derivable nets

Let Π be a $jj...j$-plane of order $q^n = h^2$. When either $h+1$ or $h-1$ divide $(q^n-1)/(q-1)$, then the subgroup of Γ of order $h+1$ or $h-1$ defines derivable nets on π. These two types of derivable nets almost never happen simultaneously. Actually,

a) $h+1$ divides $(q^n-1)/(q-1)$ only when n is even.

b) $h-1$ divides $(q^n-1)/(q-1)$ when either

  i. $q = 2$ or 3, n is even or

  ii. n is odd and $q = 4$.

Other technical results

1) A $jj \cdots j$-plane admitting a collineation switching $(x = 0)$ and $(y = 0)$ is André.

2) The transposed plane of a $jj \cdots j$-plane Π is a $jj \cdots j$-plane as well. In some cases, Π is isomorphic to its transposed plane. However, $jj \cdots j$-planes are not symplectic.

3) Replaced $jj \cdots j$-planes admit cyclic homology groups of order $(q^n - 1)/(q - 1)$, this forces them to be non-symplectic. They have also shown to be either André or new. Finally, their transposed planes are also new.

4) Derived planes are still being investigated. However, if the full collineation group of a derived $jj \cdots j$-plane Π is inherited from the original plane it was derived from, then Π is new.

5) Let π be a symplectic translation plane of order q^n and kernel isomorphic to $GF(q)$. Then any affine homology of π must have order dividing $q-1$. Thus, $j \cdots jj$-planes are not symplectic.

Before a radical change of topic... questions?

Veronese and Segre varieties

The Veronesean variety of all quadrics of $PG(n, K)$, $n \geq 1$, is the variety

$$\mathcal{V}_n = \{(x_0^2, x_1^2, \ldots, x_n^2, x_0x_1, \ldots, x_0x_n, x_1x_2, \ldots, x_{n-1}x_n);\ (x_0, x_1, \ldots, x_n) \in PG(n, K)\}$$

of $PG(N, K)$ with $N = n(n+3)/2$.

Consider the projective space $PG(n, K)$ with $n \geq 1$ and let η be a bijection between $\{0, 1, \ldots, n\} \times \{0, 1, \ldots, n\}$ and $\{0, 1, \ldots, m\}$, with $m + 1 = (n+1)^2$.

The Segre variety is the subset of $PG(m, K)$ given by:

$$\mathcal{S}_{n,n} = \left\{(x_0, x_1, \ldots, x_m);\ x_{\eta(i_1,i_2)} = x_{i_1}^{(1)} x_{i_2}^{(2)} \ \textit{with}\ (x_0^{(i)}, x_1^{(i)}, \ldots, x_{n_i}^{(i)}) \in PG(n, K)\right\}$$

A few properties of $\mathcal{V}_n$ and $\mathcal{S}_{n,n}$

1. There exist a bijection $\xi : PG(n, K) \longrightarrow \mathcal{V}_n$.

2. The quadrics of $PG(n, K)$ are mapped by ξ onto all hyperplane sections of $\mathcal{V}_n$.

3. $\mathcal{V}_1$ is a conic of $PG(2, K)$.

4. There is a bijection $\delta : PG(n, K) \times PG(n, K) \longrightarrow \mathcal{S}_{n,n}$.

5. $\mathcal{S}_{n,n} \cap \Pi_{n(n+3)/2} = \mathcal{V}_n$.

Generalizing flocks of $Q^+(3,q)$

Let $Q^+(3,q)$ denote the hyperbolic quadric of $PG(3,q)$, q any prime power. A flock of $Q^+(3,q)$ is a partition of the quadric in $q+1$ irreducible conics. A flock is linear if all the planes of the conics of the flock contain a common line.

As $Q^+(3,q)$ is the smallest Segre variety, a conic in $PG(2,q)$ is the smallest Veronesean, then we can try to extend the notion of flock to the Segre variety $S_{n,n}$ by using Veroneseans.

Given the orders of $\mathcal{S}_{n,n}$ $(|PG(n,q)|^2)$ and $\mathcal{V}_n$ $(|PG(n,q)|)$, then a partition of $\mathcal{S}_{n,n}$ into $\mathcal{V}_n$'s has to be done using $(q^n-1)/(q-1)$ Veroneseans.

Flat flocks

A flock of $\mathcal{S}_{n,n}$ is a partition of it into $(q^n-1)/(q-1)$ caps of size $(q^n-1)/(q-1)$.

If the caps are Veronesean varieties obtained as sections of $\mathcal{S}_{n,n}$ by linear subspaces of the projective space $PG(n^2+2n,q)$ in which $\mathcal{S}_{n,n}$ resides, then the flock is called a flat flock.

The flat flock is linear if all the subspaces of its Veronesean members share an n-dimensional subspace of $PG(n^2+2n,q)$.

As in the $Q^+(3,q)$ case (equivalent to planes having reguli sharing two lines), we want to relate flat flocks with a class of planes.

(A,B)-regular spreads and flat flocks

Definition Let A and B be two distinct members of a spread S of $PG(2n-1,q)$. We say S is (A,B)-regular if for every component $C \in S \setminus (A,B)$, the regulus generated by $\{A,B,C\}$ is contained in S.

Theorem Flat flocks of $\mathcal{S}_{n,n}$ are equivalent to (A,B)-regular spreads in $PG(2n-1,q)$. Moreover, the Veronese varieties of the partition correspond to $GF(q)$-reguli (reguli with $q+1$ lines).

Hyperbolic covers and flat flocks

Let S be a spread in $PG(2n-1,q)$. A "regulus hyperbolic cover of order q" of S is a set of $(q^n-1)/(q-1)$ $GF(q)$-reguli that share two components of S and whose union is S.

Theorem Flat flocks of $\mathcal{S}_{n,n}$ are equivalent to translation planes of order q^n that admit a regulus hyperbolic cover.

Some examples of flat flocks have been found. They are related to planes that are Desarguesian, semifield, nearfield or André.

Another homology group of $jj...j$-planes

There is a second homology group of order $q-1$ that is induced by

$$\Omega = \left\{ \begin{bmatrix} \Delta_M^{-1} & 0 \\ 0 & M \end{bmatrix} \in G;\ M = rId,\ r \in GF(q)^* \right\}$$

As a matter of fact, the group is

$$\Lambda = \left\{ \begin{bmatrix} r^{-1}\Delta_M^{-1} & 0 \\ 0 & Id \end{bmatrix} \in G;\ M = rId,\ r \in GF(q)^* \right\}$$

Note that Λ is $((x=0),(0))$-homology group.

The component orbits union the axis and coaxis are $\mathbb{F}_q$-reguli. Hence, we have a regulus hyperbolic cover.

Flat flocks induced by $jj...j$-planes

Since $jj...j$-planes admit a cyclic homology group of order $q-1$, then every $jj...j$-plane of order q^n induces a flat flock. Also, if $q-1$ divides $(q^n-1)/(q-1)$, then replaced $jj\cdots j$-planes induce flat flocks as well.