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Translation planes admitting a linear Abelian group of order $(q+1)^2$.

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Abstract. Translation planes of order q^2 and spread in $PG(3,q)$, where q is an odd prime power and q^2-1 has a p-primitive divisor, that admit a linear Abelian group of order $(q+1)^2$ containing at most three kernel homologies are shown to be associated to flocks of quadratic cones.

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This article is dedicated to Norm Johnson on his 70^{th} birthday.

1 Introduction

In a series of papers, that span more than fifteen years, translation planes of order q^2 with spread in $PG(3, q)$ that admit a linear collineation group G of order $q(q+1)$ were completely classified as associated to conical flocks planes. It is also known that translation planes of order q^2, with spread in $PG(3, q)$, that admit a cyclic homology group of order $q + 1$ are equivalent to conical flocks planes. Moreover, in this situation is possible to show that the full collineation group of the translation plane that admit the cyclic homology of order $q + 1$ also admits a group of order $(q + 1)^2$. In the spirit of this idea, it is possible to argue that the translation planes that admit a group of order $(q + 1)^2$ may also be associated with conical flocks planes.

We will use standard notation and results found in the literature on finite translation planes and/or flocks of quadratic cones. More details may be found in [11, 2, 10]. In particular , we will use André's [1] theory of translation planes and spreads of vector spaces. A collineation of a translation plane π is a one-to-one mapping of the points onto the points of π that preserves incidence. The collineation group of π is denoted $Aut(\pi)$, and the stabilizer of 0 is called the translation complement of π. If $\Psi \in Aut(\pi)$ fixes a line l pointwise and all the lines through a point P setwise, then Ψ is called a perspectivity of π, if $P \in l$ then Ψ is called and elation, otherwise it is called a homology. In either case, P is called the center of Ψ and l is called the axis of Ψ.

Theorem 1. *[Johnson, [6]] Translation planes with spreads in $PG(3, q)$ admitting cyclic affine homology groups of order $q + 1$ are equivalent to flocks of quadratic cones.*

Theorem 2 (Johnson [7])**.** *Let V be a vector space of dimension $2n$ over a field $F \cong GF(q)$, for $q = p^r$, p a prime. Assume that a collineation $\sigma \in GL(2n, q)$ has order dividing $q^n - 1$ but not dividing $q^t - 1$ for $t < n$. If σ fixes at least three mutually disjoint n-dimensional F-subspaces then there is an associated Desarguesian spread Σ admitting σ as a kernel homology. Furthermore, the* *normalizer of $< \sigma >$ is a collineation group of Σ. Let us call Σ an 'Ostrom phantom'.*

The problem we will study in this paper is:

Problem 1 (Il problema Abeliano rosso)**.** *Determine the translation planes π of order q^2 with spread in $PG(3, q)$ that admit an Abelian collineation group G of order $(q + 1)^2$ in $GL(4, q)$.*

A conjecture regarding this problem says that planes such as those described above must be associated to a flock of a quadratic cone. This comes after a series of papers (see for example [3, 4, 5, 8, 9]) that completely classified translation planes of order q^2 with spread in $PG(3, q)$ admitting a collineation group $G \subset GL(4, q)$ of order $q(q + 1)$. Such translation planes turned out to be conical flocks planes or derived conical flocks planes, except in a few sporadic cases, see [3] for more details. Also, it was shown that the group G is solvable and that it has a subgroup H of order $q + 1$ that normalizes an elation subgroup E of order q. Moreover, when G fixes two components of π there is an Ostrom phantom Σ associated to π, G is in $GL(2, q^2)$, and H fixes at least two components of π (one being the elation axis). It follows that H and G fix a regulus of the flock's plane.

Theorem 1 implies that if a translation plane Π with spread in $PG(3, q)$ admits a regulus inducing affine homology H_1 of order $q + 1$ in the translation complement, for example H_1 cyclic, then Π is equivalent to a conical flock plane $\mathcal{F}$. Now if one takes the normalizer N of H_1, then the quotient group N/H_1 acts as a collineation group of $\mathcal{F}$, permuting q reguli and fixing one of the reguli of $\mathcal{F}$. Connecting the previous paragraph with this idea we would have that N/H_1 has a subgroup of order $(q+1)$. Hence, the normalizer N should contain a subgroup of order $(q + 1)^2$. This justifies the conjecture.

Our main result follows, it will be proved as a series of results in the next section.

Theorem 3. *Let π be a translation plane of order q^2 (q an odd prime power) with spread in $PG(3, q)$ admitting a linear Abelian collineation group G of order $(q+1)^2$. Assume that G contains at most three kernel homologies and that q^2-1 admits a p-primitive divisor, then π is associated to a conical flock plane.*

2 Proof of the main theorem.

For the rest of this article we will assume the hypothesis of theorem 3. Also, S is a spread of π and u is a p-primitive divisor of $q^2 - 1$.

Lemma 1. *Any Sylow u-subgroup S_u of G fixes 2 components of π.*

Proof. Note that $u \neq 2$. Now let u^{2a} be the maximal power of u dividing $(q+1)^2$.

Since $q^2+1 = (q+1)(q-1)+2$, then $(q^2+1,\, u^{2a}) = 1$. It follows that the action of S_u on the components of S must fix at least one component.

Now S_u acts on q^2 components of S, but since $(q^2,\, u^{2a}) = 1$ then S_u must fix a second component. $\boxed{QED}$

Lemma 2. *Suppose an element $g \in S_u$ fixes a non-zero point in a component L that is being fixed by S_u. Then g is an affine homology with axis L.*

Proof. Since g is linear, under the hypothesis given we have that g must fix a 1-dimensional $GF(q)$-subspace A of L. Now using that $(q, u^{2a}) = 1$ we get that A has a 1-dimensional Maschke complement B.

Now recall that the order of g is a power of u, and that the number of non-zero elements in A (and B) is $q-1$. So, since $(q-1, u^t) = 1$ for any integer t, we have that g must fix a point in A (and B), and thus g must fix A and B pointwise. Hence, g fixes the component L pointwise. $\boxed{QED}$

Now we change basis, if necessary, to get the two components that are fixed by S_u to be $x = 0$ and $y = 0$. Then we consider S_u acting on the 1-dimensional subspaces of $x = 0$. Since the order of S_u is u^{2a}, and it is acting on a set with $q+1 = u^a r$ elements, where $(r, u) = 1$ and $u^{2a} > u^a$, then the stabilizer of at least one of these 1-dimensional subspaces must be non-trivial. Using the 'Maschke argument' used in the proof of the previous lemma we can assure that there is a subgroup of S_u fixing $x = 0$ pointwise, call $H^{(u)}_{x=0}$ to the largest such a subgroup. Similarly, $H^{(u)}_{y=0}$ is the largest subgroup of S_u fixing every point in $y = 0$. Moreover, they are normal in S_u and, by lemma 2, homology groups.

Lemma 3. *$H^{(u)}_{x=0}$ and $H^{(u)}_{y=0}$ are cyclic. $S_u = H^{(u)}_{x=0} \oplus H^{(u)}_{y=0}$.*

Proof. Since homology groups are Frobenius complements (see [12], for example), and Frobenius complements have cyclic odd-order Sylow subgroups, then both $H^{(u)}_{x=0}$ and $H^{(u)}_{y=0}$ are cyclic.

If we look at the orbit equation of the action of S_u on the 1-dimensional subspaces of $x = 0$ (let's call them p_i's) we get

$$u^a r = q + 1 = \sum \frac{u^{2a}}{|Stab(p_i)|}$$

where the sum considers only one p_i per orbit under S_u and $(r, u) = 1$. We notice that none of the summands can equal one because S_u cannot contain nontrivial elements that are homologies with two different axes. Also, if all the stabilizers contain less than u^a elements, then $(r, u) \neq 1$. It follows that at least one of the stabilizers has at least u^a elements. Since any element fixing a 1-dimensional subspace of $x = 0$ fixes $x = 0$ pointwise, then all stabilizers have at least u^a elements. It follows that $\left|H^{(u)}_{x=0}\right| \geq u^a$ and, similarly, $\left|H^{(u)}_{y=0}\right| \geq u^a$. Hence, $|H^{(u)}_{x=0} \cap H^{(u)}_{y=0}| = 1$ implies $S_u = H^{(u)}_{x=0} \oplus H^{(u)}_{y=0}$. $\boxed{QED}$

Remark 1. *Note that $H^{(u)}_{x=0}$ and $H^{(u)}_{y=0}$ commuting implies that they are symmetric homology groups.*

Also, the previous three lemmas are valid even when G is not Abelian.

Theorem 4. *There is $g \in S_u$ of order u that fixes 3 components of S (two of them being $x = 0$ and $y = 0$). Furthermore, there is an Ostrom phantom Σ induced by g, and $G \leq \Gamma L(2, q^2)$.*

Proof. We know $H^{(u)}_{x=0}$ is an affine homology group with axis $x = 0$ and coaxis $y = 0$ that acts on the remaining $q^2 - 1$ components of the given spread producing $(q^2 - 1)/u^a$ orbits. Note that $H^{(u)}_{y=0}$ acts on these orbits and that, since $\left(u^a, (q^2 - 1)/u^a\right) = 1$ then $H^{(u)}_{y=0}$ fixes at least one of them, call it M. Then, we can consider S_u acting on M. The orbit equation of this action is:

$$u^a = \sum \frac{u^{2a}}{|Stab(l_i)|}$$

where the sum is on the components of M, one l_i per orbit under S_u.

It is clear that none of the stabilizers can be trivial. In this way we obtain an element $g \in S_u$ of order u that fixes some l_i, $x = 0$ and $y = 0$. This element g satisfies the hypothesis of theorem 2, and thus there is an Ostrom phantom Σ and the normalizer of $< g >$ in $GL(4, q)$ is a collineation group of Σ. Since G is Abelian then it is a collineation group of Σ. $\boxed{QED}$

Lemma 4. *G has a subgroup of order $(q+1)^2/4$ which is the direct sum of two cyclic symmetric affine homology groups of order $(q + 1)/2$. Their axes/coaxes are $x = 0$ and $y = 0$.*

Proof. Let us denote by $(q + 1)_2$ the maximal power of 2 in $q + 1$, and by S_2 the 2-Sylow subgroup of G, which has order $(q + 1)_2$. Since G is Abelian, then S_2 acts on the fixed points of $H^{(u)}_{x=0}$, which forces S_2 to fix $x = 0$. Similarly, S_2 fixes $y = 0$.

Now, consider the restriction of S_2 to $x = 0$, this group is isomorphic to $S_2/(S_2)_{x=0}$, where $(S_2)_{x=0}$ is the subgroup of S_2 that fixes $x = 0$ pointwise.

But, also $S_2/(S_2)_{x=0}$ is a subgroup of $PGL(2,\, q)$, since it acts on $y = 0$, which can be regarded as a Desarguesian plane. It follows that

$$\left|\frac{S_2}{(S_2)_{x=0}}\right| = \frac{(q+1)_2^2}{|(S_2)_{x=0}|}$$

divides

$$|PGL(2,q)|_2 = (q-1)_2(q+1)_2 = 2(q+1)_2.$$

From that we get that

$$\frac{(q+1)_2}{2} \,\Bigg|\, |(S_2)_{x=0}| \,.$$

Now consider $H < G$ of order $(q+1)_{odd}^2 = (q+1)^2/(q+1)_2$. Using the same arguments used with $(S_2)_{x=0}$ we obtain that

$$(q+1)_{odd} \mid |H_{x=0}| \,.$$

Hence, the subgroup $G_{x=0}$ of G that fixes $x = 0$ pointwise has order a multiple of

$$|(S_2)_{x=0}|\,|H_{x=0}| = \frac{(q+1)_2}{2}(q+1)_{odd} = \frac{(q+1)}{2}$$

Using the same argument with $y = 0$ we obtain that $G_{x=0} \times G_{y=0}$ is the desired subgroup of G. $\boxed{QED}$

Corollary 1. *Let $G_{x=0}$ and $G_{y=0}$ be the cyclic homology groups of order $(q+1)/2$ found in lemma 4. Then, after a change of basis if necessary, $G = G_1 \times G_2$ with*

(1) $G_{x=0} < G_1 \cong \mathbb{Z}_{q+1}$ and $G_{y=0} < G_2 \cong \mathbb{Z}_{q+1}$,

(2) $G_{x=0} < G_1 \cong \mathbb{Z}_{2(q+1)}$, $G_{y=0} = G_2 \cong \mathbb{Z}_{\frac{q+1}{2}}$, and $(q+1)/2$ is odd, or

(3) $G \cong \mathbb{Z}_{2(q+1)} \times \mathbb{Z}_{\frac{q+1}{2}}$, $(q+1)/2$ is even, and at least one of $G_{x=0}$ or $G_{y=0}$ is a subgroup of one factor of G.

Proof. If $(q+1)/2$ is odd, then the 2-Sylow subgroup of G intersects $G_{x=0} \times G_{y=0}$ trivially, and thus we are done.

If $(q+1)/2$ is even then having G to be the direct product of three or more factors would force Σ to admit an elementary Abelian group of order 8 or 16, this is a contradiction. It follows that either $G \cong \mathbb{Z}_{q+1} \times \mathbb{Z}_{q+1}$ or $G \cong \mathbb{Z}_{2(q+1)} \times \mathbb{Z}_{\frac{q+1}{2}}$.

In the former case we break $G_{x=0} \times G_{y=0}$ into the direct product of its 2-Sylow subgroup H_2 with its complement. The idea used in the case $(q+1)/2$ odd is applicable to the complement of the H_2, thus we will look at this group.

Assume that $H_2 \cong \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}$ is a subgroup of $G_2 \cong \mathbb{Z}_{2^{n+1}} \times \mathbb{Z}_{2^{n+1}}$ (the 2-Sylow subgroup of G). An element in G_2 of order 2^n looks like $(\overline{a}, \overline{b})$ where both a and b are congruent to 2 modulo 4, it follows that $(\overline{a}, \overline{b}) = 2(\overline{a/2}, \overline{b/2})$. Hence, each of the $\mathbb{Z}_{2^n}$'s in H_2 is contained in some $\mathbb{Z}_{2^{n+1}}$ in G_2, thus a change of generators of G_2 implies that we can consider each of the factors of H_2 to be contained in one of the factors of G_2, which is what we wanted.

If $G \cong \mathbb{Z}_{2(q+1)} \times \mathbb{Z}_{\frac{q+1}{2}}$ and $(q+1)/2$ we break again $G_{x=0} \times G_{y=0}$ into the direct product of its 2-Sylow subgroup H_2 with its complement. In this case, an element of order 2^n can be either four times and element in G, twice but not four times an element in G, or not twice an element in G. If a generator of one of the factors of H_2 is either the first or third case, then we assure that one of the factors of $G_{x=0} \times G_{y=0}$ is a subgroup of one of the factors of G.

Let us assume that both of the generators of the factors of H_2 are twice but not four times an element of G. These elements look like $(\overline{8a+4}, \overline{4b+2})$ and $(\overline{8\alpha+4}, \overline{4\beta+2})$, where $a, b, \alpha, \beta \in \mathbb{Z}$. However,

$$2^{n-1}(\overline{8a+4}, \overline{4b+2}) = 2^{n-1}(\overline{8\alpha+4}, \overline{4\beta+2}) = (\overline{2^{n+1}}, \overline{0})$$

and thus the groups these elements generate intersect non-trivially. That is a contradiction. $\boxed{QED}$

Lemma 5. *The elements in G have the form*

$$(x, y) \longrightarrow (xa, yb)$$

where $a, b \in GF(q^2)$. In particular, $G < GL(2, q^2)$.

Proof. We know there exists an Ostrom phantom Σ associated to π, and that G is a collineation group of Σ that fixes $x = 0$ and $y = 0$. Then, a generic element $g \in G$ looks like

$$g : (x, y) \longrightarrow (x^\sigma a, y^\sigma b)$$

where σ is 1 or q, and $a, b \in GF(q^2)^*$.

We also know that $H_{y=0}^{(u)}$ is a cyclic homology group of π, thus it is generated by

$$\tau : (x, y) \longrightarrow (x, yc)$$

where $|c|$ must divide $(q+1)_{odd}$, which forces $c^\sigma = c$. Using that G is Abelian we check $g^{-1}\tau^{-1}g\tau$ for $g \in G$, as above, to obtain that $\sigma = 1$ for all $g \in G$. $\boxed{QED}$

Theorem 5. *$G_{x=0} < G_1$ and $G_{y=0} < G_2$.*

Proof. We just need to show that in the case $G_1 \cong \mathbb{Z}_{2(q+1)}$, $G_2 \cong \mathbb{Z}_{(q+1)/2}$, and $(q+1)/2$ even, it is true that $G_{x=0} \subset G_1$ if and only if $G_{y=0} = G_2$.

Assume that $G_{x=0} \subset G_1 \cong \mathbb{Z}_{2(q+1)}$. Using the previous lemma we can assume that $G_1 = \langle f : (x,y) \mapsto (ax, by) \rangle$ and $G_{x=0} = \langle f^4 : (x,y) \mapsto (a^4x, y) \rangle$, where $|a| = 2(q+1)$, and $|b|$ is divisible by 4. Similarly, $G_2 = \langle g : (x,y) \mapsto (cx, dy) \rangle$ where $lcm(|c|, |d|) = (q+1)/2$.

Recall that $a, b, c, d \in GF(q^2)$ and note that that a^4 has order $(q+1)/2$, thus $c = a^{4i}$ for some positive integer i. So, if the order of d were a proper divisor of $(q+1)/2$ then

$$e \neq g^{|d|} : (x,y) \mapsto (c^{|d|}, y)$$

which is $f^{4|d|} \in G_1$, that is a contradiction. Hence, $|d| = (q+1)/2$, and thus the element $f^{-4i}g$ generates $G_{x=0}$.

Now assume that $G_{y=0} = G_2 \cong \mathbb{Z}_{(q+1)/2}$. Using the previous lemma we can assume that $G_2 = \langle g : (x,y) \mapsto (x, by) \rangle$, where $|b| = (q+1)/2$. We can also say that G_1 is generated by $f : (x,y) \mapsto (\alpha x, \beta y)$, where $\alpha, \beta \in GF(q^2)$ and $lcm(|\alpha|, |\beta|) = 2(q+1)$.

Note that $f^4 : (x,y) \mapsto (\alpha^4 x, \beta^4 y)$ has order $(q+1)/2$, and thus both α^4 and β^4 are powers of b. Hence, composing f^4 with some power of g gives us the collineation of G

$$(x,y) \mapsto (\alpha^4 x, y)$$

which is a homology of order $(q+1)/2$. It follows that G is spanned by g and $h : (x,y) \mapsto (ax, cy)$, where the order of c is divisible by 4. It is clear that $G_{y=0}$ is contained in $< h >$, which will be our new G_1. $\boxed{QED}$

We now investigate each of the cases described in corollary 1 separately. Our goal is to show that π admits a cyclic affine homology group of order $q+1$, as having this will force π to be associated to a conical flock by [6].

When $G_{x=0} < G_1 \cong \mathbb{Z}_{q+1}$ and $G_{y=0} < G_2 \cong \mathbb{Z}_{q+1}$, we represent G as follows

$$G = \langle f : (x,y) \rightarrow (x\alpha, ya) \rangle \times \langle g : (x,y) \rightarrow (xb, y\beta) \rangle$$

where a and b have order $q+1$, and α and β have order 2. Also, without loss of generality, $G_{x=0} \subset < f >$ and $G_{y=0} \subset < g >$.

Lemma 6. *G is the direct sum of two symmetric cyclic affine homology groups of order $q+1$.*

Proof. If $(q+1)/2$ is even. Note that $f^{-(q+1)/2}g$ is defined by $(x,y) \rightarrow (xb, y)$, which is a homology of order $q+1$. Similarly, $g^{-(q+1)/2}f$ is a homology of order $q+1$ that is symmetric to $f^{-(q+1)/2}g$.

If $(q+1)/2$ is odd. Compose f and g with $(x,y) \to (-x,-y)$ to obtain

$$(x,y) \to (x,-ya) \qquad \text{and} \qquad (x,y) \to (-xb,y)$$

It is clear that these elements are affine homologies of order $q+1$ that generate G and that have symmetric axis/coaxis. $\boxed{QED}$

Now we will look at $G \cong \mathbb{Z}_{2(q+1)} \times \mathbb{Z}_{(q+1)/2}$. In this case

$$\mathbb{Z}_{2(q+1)} = \langle f : (x,y) \mapsto (x\alpha, ya) \rangle \qquad \mathbb{Z}_{(q+1)/2} = \langle g : (x,y) \to (xb,y) \rangle$$

where $a, b, \alpha \in GF(q^2)$ with $lcm[|a|, |\alpha|] = 2(q+1)$, $|b| = (q+1)/2$ and, since $f^4 \in G_{y=0}$, $|\alpha|$ divides 4.

Theorem 6. *G admits a cyclic homology group of order $q+1$.*

Proof. If $|\alpha| = 1$ or 2 then $f^2 \in G_{y=0}$ and thus the plane admits a cyclic homology group of order $q+1$.

If $|\alpha| = 4$ and $(q+1)/2$ is even, then $\alpha^2 = -1 = b^{(q+1)/4}$. It follows that $g^{(q+1)/4} f^2 \in G_{y=0}$ and it has order $q+1$.

Now assume that $|\alpha| = 4$ and $(q+1)/2$ is odd. In the case of $|a| = (q+1)/2$, then $f^{(q+1)/2} \in G_{x=0}$. It follows that $f^{(q+1)/2} g \in G_{x=0}$ and has order $2(q+1)$. Similarly, if $|a| = q+1$, then $f^{q+1} \in G_{x=0}$, and thus $f^{q+1} g \in G_{x=0}$ and has order $q+1$. In either case the plane admits a cyclic homology group of order $q+1$.

Finally, consider $|\alpha| = 4$, $|a| = 2(q+1)$, and $(q+1)/2$ odd. Note that $| < \alpha b > | = 2(q+1)$. So, there is a positive integer i such that either $a = \alpha b^i$ or $a^{-1} = \alpha b^i$.

In the first case, $f^{(q+1)/2}$ is a kernel homology of order 4, which contradicts our hypothesis.

In the second case, $a^{-1} = \alpha b^i$ forces $a^{-(q+1)/2} = \alpha^{(q+1)/2} = \alpha^{\pm 1}$. We will now look at these two cases separately.

If $a^{-(q+1)/2} = \alpha$, we use $\alpha^{-1} = ab^i$ to get $\alpha^{-(q+1)/2} = a^{(q+1)/2}$. It follows that $\alpha^{(q+1)/2} = \alpha$. All this implies that $f^{(q+1)/2}$ is defined by

$$f^{(q+1)/2} : (x,y) \mapsto (x\alpha, y\alpha^{-1})$$

Since $(q+1)/2$ odd then $q \equiv 1 \pmod 3$, thus $\alpha \in GF(q)$. It follows that the map

$$\sigma : (x,y) \mapsto (x\alpha, y\alpha)$$

is a kernel homology, which put together with $f^{(q+1)/2}$ yields

$$\sigma f^{(q+1)/2} : (x,y) \mapsto (-x,y)$$

Hence, the map $\sigma f^{(q+1)/2} g$ is an element of $G_{x=0}$ of order $q+1$.

Finally, if $a^{(q+1)/2} = \alpha$, just as we did above, we get $\alpha^{(q+1)/2} = \alpha^{-1}$ and thus

$$f^{(q+1)/2} : (x, y) \mapsto (x\alpha^{-1}, y\alpha)$$

The kernel homology $\tau : (x, y) \mapsto (x\alpha^{-1}, y\alpha^{-1})$ finishes the proof. $\boxed{QED}$

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