Symplectic Geometry Over Finite Fields

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Symplectic Forms

Definition

Let V be a $2n$-dimensional vector space over a field K. A symplectic form in V is a bilinear form

$$\mathcal{B} : V \times V \to K$$

such that $\mathcal{B}(v, v) = 0$ for all $v \in V$ (i.e. $\mathcal{B}$ is alternating).

We say that $(V, \mathcal{B})$ is a symplectic space.

Symplectic Forms

- $\mathcal{B}$ may be represented using some $B \in M_{2n}(K)$ in the following way
$$\mathcal{B}(v, w) = vBw^t$$

- We will say that the form $\mathcal{B}$ is non-degenerate if the matrix B has rank $2n$ (i.e. B is non-singular).

Standard Symplectic Form

- All non-degenerate symplectic forms of V are equivalent.
- Moreover, it is possible to find a basis of V such that.

$$B = \left[\begin{array}{c|c} & I \\ \hline -I & \end{array}\right]$$

- This basis is called a symplectic basis of V.

Totally Isotropic Subspaces

- A subspace S of V such that $\mathcal{B}(v, w) = 0$ for all $v, w \in S$ is said to be a totally isotropic subspace of V (relative to $\mathcal{B}$).
- Maximal totally isotropic subspaces of V have dimension n.

All n-dimensional subspaces of V that intersect $< e_{n+1}, \cdots, e_{2n} >$ trivially can be represented as

$$S = \{(x_1, \cdots, x_n, y_1, \cdots, y_n) \in V \ ; \ (y_1, \cdots, y_n) = (x_1, \cdots, x_n)M\}$$

for some $M \in M_n(K)$. We will say that $S = (y = xM)$.

An Example

For a fixed $\alpha \in K$, consider the subspace

$$S = (y = x\alpha)$$

For B, the "standard" form given above, and $x, a \in K^n$

$$\begin{aligned} [x, x\alpha]\ B\ [a, a\alpha]^t &= -x\alpha a^t + x(a\alpha)^t \\ &= 0 \end{aligned}$$

So, S is a maximal totally isotropic subspace of V relative to $\mathcal{B}$.

Generalization

Lemma

An n dimensional subspace S of V of the form

$$S = (y = xM)$$

is a maximal totally isotropic subspace of V relative to $\mathcal{B}$, if and only if, M is symmetric.

Note that the subspace $< e_{n+1}, \cdots, e_{2n} >$ is also a maximal totally isotropic subspace of V relative to $\mathcal{B}$.

Symplectic Spreads of V

- If $dim_K(V) = 2n$, then a spread of V is collection of n-dimensional subspaces of V that pairwise intersect trivially and cover V.
- The elements of S are called components of S. Note that the direct sum of any two components is V.
- A spread formed by maximal totally isotropic subspaces is called a symplectic spread.

Symplectic Graphs

- In a $2n$-dimensional symplectic space $(V, \mathcal{B})$ define a graph G having vertices all 1-dimensional subspaces of V and edges given by

$$< v > \sim < w > \Leftrightarrow < v, w > \text{ is totally isotropic}$$

- When V is a vector space over $GF(q)$, then this graphs is called $Sp(2n, q)$ (awful notation!)

- $Sp(2n, q)$ is a strongly regular graph with parameters

$$\left(\frac{q^{2n} - 1}{q - 1}, q^{2n-1}, q^{2n-2}(q-1), q^{2n-2}(q-1) \right)$$

Symplectic Graphs

- A graph G is said to be k-partite if the vertices of G partition in k sets $\{X_i\}$ such that there are no edges of G among vertices in the same X_i.
- $Sp(2n, q)$ is $(q^n + 1)$-partite.

  A symplectic spread provides the partition of the vertices of G.
- Non-isomorphic spreads yield non-isomorphic graphs?
- $\chi(Sp(2n, q)) = q^n + 1$.