Symplectic Geometry Over Finite Fields 2

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Non-Singular Symplectic Forms over $GF(q)$

Definition

Let V be a 4-dimensional vector space over the finite field $GF(q)$. We say that $(V, \mathcal{B})$ is a (non-singular) symplectic space if it admits a bilinear form

$$\mathcal{B} : V \times V \rightarrow GF(q)$$

defined by some $B \in GL(4, q)$ as follows

$$\mathcal{B}(v, w) = vBw^t$$

and such that $\mathcal{B}(v, v) = 0$ for all $v \in V$.

Symplectic Forms

- It is known that such a form looks like

$$B = \left[\begin{array}{cc|cc} & & 1 & \\ & & & 1 \\ \hline -1 & & & \\ & -1 & & \end{array}\right]$$

  after a change of basis in V.
- A subspace S of V such that $\mathcal{B}(v, w) = 0$ for all $v, w \in S$ is said to be a totally isotropic subspace of V (relative to $\mathcal{B}$).
- Maximal totally isotropic subspaces of V are 2-dimensional.

2-Dimensional Subspaces of V

- We will write V as

$$V = \{(x, y) \ ; \ x = (x_1, x_2), y = (y_1, y_2) \in GF(q)^2\}$$

- The 2-dimensional subspace of V

$$S = \{(0, 0, y_1, y_2) \ ; \ y_1, y_2 \in GF(q)\}$$

  will be denoted by $(x = 0)$.

- All 2-dimensional subspaces of V that intersect $(x = 0)$ trivially can be represented as

$$S = \{(x, y) \in V \ ; \ (y_1, y_2) = (x_1, x_2)M\}$$

  for some $M \in M_2(K)$. We will denote S by $(y = xM)$.

Maximal Totally Isotropic Subspaces of V

- Recall that a maximal totally isotropic subspace of V is a 2-dimensional subspace S such that $\mathcal{B}(v, w) = 0$ for all $v, w \in S$.
- $(x = 0)$ is totally isotropic with respect to the 'standard' symplectic form $\mathcal{B}$.
- An 2-dimensional subspace of V of the form $(y = xM)$ is a maximal totally isotropic subspace of V relative to $\mathcal{B}$, if and only if, M is symmetric.

$PG(3, q)$

The 3-dimensional projective geometry, or $PG(3, q)$, consists of points, lines, planes, and a rule (called incidence) that determines when an object is contained in another.

- The points of $PG(3, q)$ are the 1-dimensional subspaces of V.
- The lines of $PG(3, q)$ are the 2-dimensional subspaces of V.
- The planes of $PG(3, q)$ are the 3-dimensional subspaces of V.
- The incidence is given by regular set-theoretic containment.

This is a non-Euclidean space.

Symplectic Spreads of $PG(3, q)$ (or V)

- A spread $\mathcal{S}$ of $PG(3, q)$ is collection of $q^2 + 1$ lines that are pairwise disjoint (and that partition the points of $PG(3, q)$).
- Equivalently. A spread $\mathcal{S}$ of V is a collection of $q^2 + 1$ 2-dimensional subspaces of V that intersect trivially (and that cover V.)
- The elements of $\mathcal{S}$ are called components of $\mathcal{S}$. Note that the direct sum of any two components is equal to V.
- A spread formed by maximal totally isotropic subspaces of V is called a symplectic spread.

Generalized Quadrangles

- A generalized quadrangle of order (s, t) (or an $(s, t) - GQ$) is a triple $(\mathcal{P}, \mathcal{L}, \mathcal{I})$, where $\mathcal{P}$ is a set of points, $\mathcal{L}$ is a set of lines, and $\mathcal{I}$ is an incidence rule, such that

  1. there are exactly $s + 1$ points per line,
  2. there are exactly $t + 1$ lines per point,
  3. for every point p not on a line L, there is a unique line M and a unique point q, such that p is on M, and q is on $M \cap L$.

- The geometry formed by the points of $PG(3, q)$ and all the symplectic lines of $PG(3, q)$ is a $(q, q) - GQ$ called $\mathcal{W}(q)$.

$\mathcal{W}(q)$, Symplectic Spreads, And Duality

- A spread of a GQ T is a collection of lines of T that partition the points of T.
- $\mathcal{S}$ is a symplectic spread of $PG(3, q)$, if and only if, $\mathcal{S}$ is a spread of $\mathcal{W}(q)$.
- If T is an $(s, t) - GQ$, then by relabeling points as lines and lines as points we obtain a $(t, s) - GQ$ T', the dual of T.
- The dual of $\mathcal{W}(q)$ is $\mathcal{W}(q)$, if and only if, q is even.

Ovoids of $PG(3, q)$

- The dual of a spread of $\mathcal{W}(q)$ is a collection of $q^2 + 1$ points such that no line of $\mathcal{W}(q)$ goes through an three of them. This is called an Ovoid of $\mathcal{W}(q)$.
- An ovoid of $PG(3, q)$ is a set of $q^2 + 1$ points of $PG(3, q)$ such that no three of them are in a line of $PG(3, q)$.
- J.A. Thas showed that an ovoid of $\mathcal{W}(q)$ is an ovoid of $PG(3, q)$.

Summary

$$\begin{array}{ccc} \textit{Sympl. Spreads of } PG(3,q) & = & \textit{Spreads of } \mathcal{W}(q) \\ \textit{Klein Correspondence} \big\uparrow & & \big\downarrow \textit{dualize} \\ \textit{Ovoids of } PG(3,q) & = & \textit{Ovoids of } \mathcal{W}(q) \end{array}$$