Mathematics Department Qualifying Exam : Algebra Fall 2002

Part A. Do **five** of the following eight problems :

1. Let S be a set, and let $\mathcal{P}(S) = \{A|\ A \subset S\}$ be the collection of all subsets of S. Define a relation $\sim$ on $\mathcal{P}(S)$ by letting $A \sim B$ if and only if there is a one-to-one correspondence from A to B. Prove that $\sim$ is an equivalence relation.

2. Let G be a group and let $f : G \to G$ be defined by $f(g) = g^{-1}$ for all $g \in G$. Under what conditions is f a group homomorphism? Justify your answer.

3. Let $f : G \to H$ be a group homomorphism. Prove that $ker(f)$ is a normal subgroup of G.

4. Show that in a finite cyclic group of order n with identity element e, the equation $x^m = e$ has exactly m solutions, for each positive integer m that is a divisor of n.

5. Let G be any group with no proper nontrivial subgroups, and assume the order of G is greater than 1. Prove that G is cyclic of order p for some prime p.

6. (a) Let R be a commutative ring such that $a^2 = a$ for each $a \in R$. Prove that $a + a = 0$ for each $a \in R$.
   (b) Prove that $(a + b)(a - b) = a^2 - b^2$ for all a, b in a ring R if and only if R is commutative.

7. Let R be the set of all continuous functions from the set of real numbers to itself. Define addition and multiplication of $f, g \in R$ by
   $$(f + g)(x) = f(x) + g(x) \qquad \text{and} \qquad (f \cdot g)(x) = f(x)g(x)$$
   for all numbers x.
   Prove that R is a commutative ring under these operations.

8. Elements a and b of a ring R are called *zero divisors* if a and b are nonzero and $ab = 0$. Prove that every finite commutative ring with no zero divisors is a field.

Part B. Solve **five** of the following eight problems :

1. (a) Show that $A = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}$ is not invertible for any choice of a and b.

   (b) If A is any matrix, show that AA^T and A^TA are both symmetric matrices.

2. Find all 3×3 diagonal matrices A that satisfy $A^2 - 3A - 4I = 0$

3. If A is an $n \times n$ diagonalizable matrix with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$, show that $\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$

4. Show that

$$< (u_1, u_2), (v_1, v_2) >= \frac{1}{4} u_1 v_1 + \frac{1}{16} u_2 v_2$$

   defines an inner product on $\mathbb{R}^2$.

5. Let P_n be the vector space of polynomials in x of degree at most n with real coefficients union the zero polynomial. Determine the dimension of the subspace of P_n consisting of all polynomials

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

   for which $a_0 = 0$.

6. Show that $\begin{bmatrix} 2 & 1 \\ 1 & -5 \end{bmatrix}$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ commute if $a - d = 7b$.

7. Suppose A is a square matrix. Suppose x is an eigenvector of A with corresponding eigenvalue λ, and y is an eigenvector of A^T with corresponding eigenvalue μ. Show that if $\lambda \neq \mu$, then $x \cdot y = 0$.

8. Let $B = \{v_a, v_2, v_3, v_4\}$ be a basis for a vector space V. Find the matrix with respect to B of the linear operator $T : V \to V$ defined by $T(v_1) = v_2$, $T(v_2) = v_3$, $T(v_3) = v_4$, and $T(v_4) = v_1$.