Algebra Qualifying Exam, Solutions Fall 2002

Part A.

1. Let S be a set, and let $\mathcal{P}(S) = \{A|\ A \subset S\}$ be the collection of all subsets of S. Define a relation $\sim$ on $\mathcal{P}(S)$ by letting $A \sim B$ if and only if there is a one-to-one correspondence from A to B. Prove that $\sim$ is an equivalence relation.

   Solution. Since the identity defines a one-to-one correspondence from a set to itself and if $f : S \to T$ defines a one-to-one correspondence then $f^{-1} : T \to S$ is also a one-to-one correspondence, then we just need to check transitivity.

   So, assume $f : S \to T$ is a one-to-one correspondence and $g : T \to U$ is also a one-to-one correspondence. Consider the map

   $$g \circ f : S \to U$$

   Since both f and g are bijective, then so is fg.

2. Let G be a group and let $f : G \to G$ be defined by $f(g) = g^{-1}$ for all $g \in G$. Under what conditions is f a group homomorphism? Justify your answer.

   Solution. If f is a group homomorphism, then for any $g, h \in G$ we get

   $$h^{-1}g^{-1} = (gh)^{-1} = f(gh) = f(g)f(h) = g^{-1}h^{-1}$$

   So, $h^{-1}g^{-1} = g^{-1}h^{-1}$ for all $g, h \in G$. Since every element in G is the inverse of an element in G, then G is Abelian.

   Clearly, if G is Abelian, then f is a homomorphism. Hence,

   $$f \text{ is a homomorphism } \Leftrightarrow G \text{ is an Abelian group}$$

3. Let $f : G \to H$ be a group homomorphism. Prove that $ker(f)$ is a normal subgroup of G.

Solution. Let us first check it is a subgroup. We know the identity of the group is there, so it is not empty. Also, for $g, h \in ker(f)$

$$f(gh) = f(g)f(h) = e \cdot e = e$$

and

$$e = f(gg^{-1}) = f(g)f(g^{-1}) = e \cdot f(g^{-1}) = f(g^{-1})$$

So, $ker(f)$ is a subgroup of G

Finally, let $x \in G$, then

$$f(xgx^{-1}) = f(x)f(g)f(x^{-1}) = f(x) \cdot e \cdot f(x)^{-1} = f(x)f(x)^{-1} = e$$

4. Show that in a finite cyclic group of order n with identity element e, the equation $x^m = e$ has exactly m solutions, for each positive integer m that is a divisor of n.

   Solution. Let $G =< g >$ and let m be a divisor of n. Then the element $h = g^{n/m} \in G$ has order m (otherwise the order of g wouldn't be n). The subgroup $< h >$ has m elements, and all of them (by Lagrange's theorem) satisfy $x^m = e$. So, we have at least m solutions for the equation $x^m = e$.

   Now consider $x = g^i \in G$ be such that $x^m = e$. Then

$$e = (g^i)^m = g^{im}$$

   So, n divides im, and thus i is a multiple of $\frac{n}{m}$. Hence $g^i \in< h >$.

5. Let G be any group with no proper nontrivial subgroups, and assume the order of G is greater than 1. Prove that G is cyclic of order p for some prime p.

   Solution. If G contains an element g of infinite order, then $\mathbb{Z} \cong< g >< G$, and thus G contains infinitely many subgroups. So, all elements of G have finite order.

   Let $g \in G$, if there is an element $h \in G$ that does not belong to $< g >$, then $< g >$ is a proper subgroup of G, a contradiction. It follows that G is a (finite) cyclic group. Hence $G \cong \mathbb{Z}_m$ for some m. We know that $\mathbb{Z}_m$ contains subgroups of order any divisor of m. It follows that G must be isomorphic to $\mathbb{Z}_p$ for some prime number p

6. (a) Let R be a commutative ring such that $a^2 = a$ for each $a \in R$. Prove that $a + a = 0$ for each $a \in R$.

   (b) Prove that $(a + b)(a - b) = a^2 - b^2$ for all a, b in a ring R if and only if R is commutative.

Solution.

(a)

$$a + a = (a + a)^2 = a^2 + 2a^2 + a^2 = a + 2a + a = 2(a + a)$$

So, $a + a = 0$.

(b) Assume the ring is commutative, then

$$(a + b)(a - b) = a^2 - ab + ba - b^2 = a^2 - ab + ab - b^2 = a^2 - b^2$$

Assume $(a + b)(a - b) = a^2 - b^2$ for all a, b in R. Since

$$(a+b)(a-b) = a^2 - ab + ba - b^2 \qquad \text{and} \qquad a^2 - b^2 = a^2 - ab + ab - b^2$$

then $a^2 - ab + ba - b^2 = a^2 - ab + ab - b^2$, which implies $ba = ab$

7. Let R be the set of all continuous functions from the set of real numbers to itself. Define addition and multiplication of $f, g \in R$ by

$$(f + g)(x) = f(x) + g(x) \qquad \text{and} \qquad (f \cdot g)(x) = f(x)g(x)$$

for all numbers x.

Prove that R is a commutative ring under these operations.

Solution. Clearly closure holds for both operations (Calculus!!). Also, the zero function is the additive identity and the identity map is the multiplicative identity of R.

All the other things needed to check for R to be a commutative ring follow trivially from the fact that $\mathbb{R}$ is a commutative ring.

8. Elements a and b of a ring R are called *zero divisors* if a and b are nonzero and $ab = 0$. Prove that every finite commutative ring with no zero divisors is a field.

Solution. In problem 8 of part A in the exam of Spring 2003 it is shown that a ring R as the one we are considering must have a one (multiplicative identity). That proof uses that, for a fixed $r \in R^*$, the function $\phi_r : R \to R$ defined by $\phi_r(x) = rx$ is bijective.

Since we know that R has a one, then there must be an $s \in R$ such that $\phi_r(s) = 1$ (ϕ is onto!). It follows that $rs = 1$, and thus r has an inverse.

Part B.

1. (a) Show that $A = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}$ is not invertible for any choice of a and b.

   (b) If A is any matrix, show that AA^T and A^TA are both symmetric matrices.

   Solution.

   (a) The determinant of A is zero no matter what a and b are.

   (b) $(AA^T)^T = (A^T)^T A^T = AA^T$.
   To show A^TA is symmetric we proceed in the same way.

2. Find all 3×3 diagonal matrices A that satisfy $A^2 - 3A - 4I = 0$

   Solution. Since A satisfies $A^2 - 3A - 4I = 0$, then the minimal polynomial of A divides $x^2 - 3x - 4 = (x-4)(x+1)$. It follows that the non-zero entries of A can only be -1 and/or 4 (using that A is diagonal). So, there are 8 possibilities for A (too much to type, so I will write that down in an odd way)

$$A = \begin{bmatrix} 1.5 \pm 2.5 & 0 & 0 \\ 0 & 1.5 \pm 2.5 & 0 \\ 0 & 0 & 1.5 \pm 2.5 \end{bmatrix}$$

3. If A is an $n \times n$ diagonalizable matrix with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$, show that $\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$

   Solution. We know $\det(AB) = \det(A)\det(B)$.

   Since A is diagonalizable, then there is a matrix P such that $A = PDP^{-1}$, where D is diagonal with and diagonal entries given by the n eigenvalues of A. Hence,

$$\det(A) = \det(PDP^{-1}) = \det(P)\det(D)\det(P^{-1}) = \det(D) = \lambda_1 \lambda_2 \cdots \lambda_n$$

4. Show that

$$< (u_1, u_2), (v_1, v_2) >= \frac{1}{4} u_1 v_1 + \frac{1}{16} u_2 v_2$$

   defines an inner product on $\mathbb{R}^2$.

Solution. Let $u = (u_1, u_2), v = (v_1, v_2) \in \mathbb{R}^2$.

$$< u, u >= \frac{1}{4}(u_1)^2 + \frac{1}{16}(u_2)^2$$

which is greater or equal to zero, and zero only when both u_1 and u_2 are zero. Also,

$$< u, v >= \frac{1}{4}u_1v_1 + \frac{1}{16}u_2v_2 = \frac{1}{4}v_1u_1 + \frac{1}{16}v_2u_2 =< v, u >$$

And finally, for $\alpha, \beta \in \mathbb{R}$ and $w = (w_1, w_2) \in \mathbb{R}^2$

$$\begin{aligned}
< \alpha u + \beta v, w > &= < (\alpha u_1 + v_1, \alpha u_2 + v_2), (w_1, w_2) > \\
&= \frac{1}{4}(\alpha u_1 + \beta v_1)w_1 + \frac{1}{16}(\alpha u_2 + \beta v_2)w_2 \\
&= \frac{1}{4}\alpha u_1 w_1 + \frac{1}{4}\beta v_1 w_1 + \frac{1}{16}\alpha u_2 w_2 + \frac{1}{16}\beta v_2 w_2 \\
&= \left(\frac{1}{4}\alpha u_1 w_1 + \frac{1}{16}\alpha u_2 w_2\right) + \left(\frac{1}{4}\beta v_1 w_1 + \frac{1}{16}\beta v_2 w_2\right) \\
&= \alpha\left(\frac{1}{4}u_1 w_1 + \frac{1}{16}u_2 w_2\right) + \beta\left(\frac{1}{4}v_1 w_1 + \frac{1}{16}v_2 w_2\right) \\
&= \alpha < u, w > +\beta < v, w >
\end{aligned}$$

5. Let P_n be the vector space of polynomials in x of degree at most n with real coefficients union the zero polynomial. Determine the dimension of the subspace of P_n consisting of all polynomials

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

for which $a_0 = 0$.

Solution. We are assuming that

$$S = \{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \in P_n;\ a_0 = 0\}$$

is a subspace (if you want to show it, consider the homomorphism $\phi : P_n \to \mathbb{R}$ defined by $\phi(p(x)) = p(0)$)

It is clear that S is spanned by

$$B = \{x, x^2, \cdots, x^n\}$$

Since B is a subset of the standard basis of $\mathbb{R}[x]$, then it must be linearly independent. It follows that the dimension of S is $n - 1$.

6. Show that $\begin{bmatrix} 2 & 1 \\ 1 & -5 \end{bmatrix}$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ commute if $a - d = 7b$.

Solution. We want the following two products to be the same

$$\begin{bmatrix} 2 & 1 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2a+c & 2b+d \\ a-5c & b-5d \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & -5 \end{bmatrix} = \begin{bmatrix} 2a+b & a-5b \\ 2c+d & c-5d \end{bmatrix}$$

Then, the two $1,1$ entries being the same implies $c = b$ (also implies by the two $2,2$ entries being the same). So, now we have just one equation

$$2b + d = a - 5b \qquad \text{or} \qquad a - d = 7b$$

7. Suppose A is a square matrix. Suppose x is an eigenvector of A with corresponding eigenvalue λ, and y is an eigenvector of A^T with corresponding eigenvalue μ. Show that if $\lambda \neq \mu$, then $x \cdot y = 0$.

Solution. This is problem 7 in part B in the exam of Spring 2008.

8. Let $B = \{v_a, v_2, v_3, v_4\}$ be a basis for a vector space V. Find the matrix with respect to B of the linear operator $T : V \to V$ defined by $T(v_1) = v_2$, $T(v_2) = v_3$, $T(v_3) = v_4$, and $T(v_4) = v_1$.

Solution. Since we are looking for the matrix of T with respect to the base B, then we have to consider the v_1's as if they were the canonical basis of, in this case, $\mathbb{R}^4$. So,

$$[T]_B = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$