Mathematics Department Qualifying Exam Fall 2005
Subject: Algebra

Part A. Do **five** of the following 8 problems.

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1. Prove that if G is any cyclic group, then G is abelian. *(Your proof must include the case where G is infinite.)*

2. Let S_n denote the group of permutations on the set $\{1, 2, \ldots, n\}$, and let $\sigma \in S_n$ be an odd permutation. Prove that σ^{-1} is an odd permutation.

3. Let N be a normal subgroup of an abelian group G. Prove that the factor group G/N is abelian.

4. Let $G = \mathbb{Q} - \{-1\}$ be the set of rational numbers except -1. For a, $b \in G$, let $a * b$ be defined by $a * b = a + b + ab$. Prove that $(G, *)$ is a group.

5. Let H be a subgroup of a group G. For a, $b \in G$, define a relation on G by letting $a \sim b$ if and only if $ab^{-1} \in H$. Prove that $\sim$ is an equivalence relation.

6. Prove that the polynomial $f(x) = x^7 - 10x^4 + 15x - 5$ is irreducible over the field of rational numbers.

7. An element a of a ring R is called a *zero divisor* if there exists a nonzero element $b \in R$ such that $ab = 0$. If a is a zero divisor of a commutative ring R and $r \in R$, prove that ar is a zero divisor.

8. Let R and S be commutative rings, and let $\varphi \colon R \to S$ be a non-trivial ring homomorphism.

   (a) Prove that the kernel of φ is an ideal of R.

   (b) If R is a field, prove that φ is one-to-one.

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Part B. Do **five** of the following 8 problems.

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1. Let $L_A\colon \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation given by $L_A(\mathbf{v}) = A\mathbf{v}$ where A is a real $n \times n$ matrix. Show that if n is odd then L has a real eigenvalue.

2. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are distinct eigenvectors corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of a matrix A, prove that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a linearly independent set.

3. Let V be a real vector space of dimension n, and suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_t\}$ is a linearly independent subset of V. Prove that there is a basis B of V such that $S \subseteq B$.

4. Let S be the set of all functions $f\colon \mathbb{R} \to \mathbb{R}$ that satisfy the differential equation
$$y'' - y' + 2y = 0.$$
Is S a real vector space? (Assume the usual operations $(f+g)(t) = f(t) + g(t)$ and $(c \cdot f)(t) = cf(t)$ where $c \in \mathbb{R}$.) Explain why or why not.

5. Let A and B be $n \times n$ matrices. Show that AB is invertible if and only if A and B are invertible.

6. Prove that a linear transformation of vector spaces $L\colon V \to W$ is one to one if and only if L maps linearly independent subsets of V to linearly independent subsets of W.

7. Let V be a finite dimensional vector space and $L : V \to W$ be a linear transformation to vector space W. Prove that $\dim(\text{kernel } L) + \dim(\text{image } L) = \dim V$.

8. Let A be an $n \times n$ matrix. Prove that A is invertible if and only if the determinant of A is non-zero.