Algebra Qualifying Exam, Solutions Fall 2005

Part A.

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1. Prove that if G is any cyclic group, then G is abelian. *(Your proof must include the case where G is infinite.)*

   Solution. Let $G = \langle a \rangle$, and let x, $y \in G$. Then $x = a^k$ and $y = a^l$ for some $k, l \in \mathbb{Z}$. Thus $xy = a^k a^l = a^{k+l} = a^l a^k = yx$.

2. Let S_n denote the group of permutations on the set $\{1, 2, \ldots, n\}$, and let $\sigma \in S_n$ be an odd permutation. Prove that σ^{-1} is an odd permutation.

   Solution. σ can be written as a product of an odd number of transpositions. Suppose
   $$\sigma = (a_1\, b_1)(a_2\, b_2) \cdots (a_{2k+1}\, b_{2k+1}).$$
   Then $\sigma^{-1} = (a_{2k+1}\, b_{2k+1}) \cdots (a_1\, b_1)$, again a product of an odd number of transpositions. Therefore σ^{-1} is odd.

3. Let N be a normal subgroup of an abelian group G. Prove that the factor group G/N is abelian.

   Solution. Let aN, $bN \in G/N$. Then $aNbN = abN = baN = bNaN$.

4. Let $G = \mathbb{Q} - \{-1\}$ be the set of rational numbers except -1. For a, $b \in G$, let $a * b$ be defined by $a * b = a + b + ab$. Prove that $(G, *)$ is a group.

   Solution.

   Closure. If a, $b \in G$, then clearly $a+b+ab \in \mathbb{Q}$. If $a+b+ab = -1$ then $a(1+b) = -(b+1)$, and hence $a = -\frac{b+1}{b+1} = -1$, a contradiction. Therefore G is closed under $*$.

   Associativity. We have
   $$\begin{aligned}(a * b) * c = (a + b + ab) * c &= (a + b + ab) + c + (a + b + ab)c \\ &= a + b + c + ab + ac + bc + abc \\ &= a + (b + c + bc) + a(b + c + bc) = a * (b + c + bc) = a * (b * c).\end{aligned}$$

   Identity. We claim that $e = 0$ is the identity element. For all $a \in G$ we have
   $$a + 0 + a \cdot 0 = a = 0 + a + 0 \cdot a.$$

   Inverse. Let $a \in G$. We claim that $a^{-1} = -\frac{a}{1+a}$, which is an element of G since $a \neq 1$. We have
   $$a + \left(-\frac{a}{1+a}\right) + a \cdot \left(-\frac{a}{1+a}\right) = \frac{a(1+a) - a - a^2}{1+a} = 0.$$

   Thus $(G, *)$ is a group.

5. Let H be a subgroup of a group G. For a, $b \in G$, define a relation on G by letting $a \sim b$ if and only if $ab^{-1} \in H$. Prove that $\sim$ is an equivalence relation.

   Solution.

   Reflexivity. Let $a \in G$. Then $aa^{-1} = e \in H$. Thus $a \sim a$.

   Symmetry. If $ab^{-1} \in H$, then $(ab^{-1})^{-1} = ba^{-1} \in H$. Thus $a \sim b \Rightarrow b \sim a$.

   Transitivity. If ab^{-1}, $bc^{-1} \in H$, then $ab^{-1}bc^{-1} = ac^{-1} \in H$. Thus $a \sim b$, $b \sim c \Rightarrow a \sim c$.

   Thus $\sim$ is an equivalence relation.

6. Prove that the polynomial $f(x) = x^7 - 10x^4 + 15x - 5$ is irreducible over the field of rational numbers.

   Solution. Using Eisenstein's Criterion with $p = 5$, we see that $5 \mid a_i$ for $0 \leq i \leq 6$ where a_i is the coefficient of x^i; also 5 does not divide the leading coefficient, and 25 does not divide the constant term. Therefore $f(x)$ is irreducible over $\mathbb{Q}$.

7. An element a of a ring R is called a *zero divisor* if there exists a nonzero element $b \in R$ such that $ab = 0$. If a is a zero divisor of a commutative ring R and $r \in R$, prove that ar is a zero divisor.

   Solution. If a is a zero divisor, then there is a nonzero element $b \in R$ such that $ab = 0$. Therefore $arb = rab = 0$. Thus ar is a zero divisor.

8. Let R and S be commutative rings, and let $\varphi\colon R \to S$ be a non-trivial ring homomorphism.

   (a) Prove that the kernel of φ is an ideal of R.

   (b) If R is a field, prove that φ is one-to-one.

   Solution.

   (a) Let k_1, $k_2 \in \ker(\varphi)$ and $r \in R$. Then $\varphi(k_1 + k_2) = \varphi(k_1) + \varphi(k_2) = 0 + 0 = 0$. Thus $k_1 + k_2 \in \ker(\varphi)$. Also $\varphi(rk_1) = \varphi(r)\varphi(k_1) = \varphi(0) \cdot 0 = 0$; therefore $rk_1 \in \ker(\varphi)$. Thus $\ker(\varphi) \lhd R$.

   (b) The only ideals of a field are $\{0\}$ and the field itself. Thus $\ker(\varphi) = \{0\}$ or $\ker(\varphi) = F$. But φ is given to be nontrivial; therefore $\ker(\varphi) \neq F$. Hence $\ker(\varphi) = \{0\}$, and φ is one-to-one.

Part B.

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1. Let $L_A\colon \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation given by $L_A(\mathbf{v}) = A\mathbf{v}$ where A is a real $n \times n$ matrix. Show that if n is odd then L has a real eigenvalue.

   Solution. Since the dimension of $\mathbb{R}^n$ is odd the characteristic polynomial of A will be of odd degree. Since every polynomial in $\mathbb{R}[x]$ of odd degree has a real root, we conclude that L_A has a real eigenvalue.

2. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are distinct eigenvectors corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of a matrix A, prove that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a linearly independent set.

   Solution. Assume that $v_1, v_2, \ldots, v_n$ form a linearly dependent set. Let k be the smallest positive integer such that $v_k \in \operatorname{span}\{v_1, v_2, \ldots, v_{k-1}\}$. Thus there are real numbers $c_1, \ldots, c_{k-1}$ such that

   $$v_k = c_1 v_1 + c_2 v_2 + \cdots + c_{k-1} v_{k-1} \tag{1}$$

   Hence,

   $$A v_k = c_1 A v_1 + c_2 A v_2 + \cdots + c_{k-1} A v_{k-1} \qquad \text{thus} \tag{2}$$

   $$\lambda_k v_k = c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \cdots + c_{k-1} \lambda_{k-1} v_{k-1} \tag{3}$$

   Now subtract equation (3) from λ_k times equation (1) to get

   $$0 = c_1(\lambda_k - \lambda_1)v_1 + c_2(\lambda_k - \lambda_2)v_2 + c_{k-1}(\lambda_k - \lambda_{k-1})v_{k-1}$$

   Since the λ_i are all distinct and the set $\{v_1, v_2, \ldots, v_{k-1}\}$ is independent, we must have that all of the c_i are zero. It follows that v_k is zero, contradicting the hypothesis that v_k is an eigenvector of A.

3. Let V be a real vector space of dimension n, and suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_t\}$ is a linearly independent subset of V. Prove that there is a basis B of V such that $S \subseteq B$.

   Solution. If $t = n$ then S is a basis. So assume that $t < n$. Let $B = \{w_1, \ldots, w_n\}$ be a basis of V. Consider the set $B^*\{v_1, v_2, \ldots, v_t, w_1, w_2, \ldots, w_n\}$. Clearly this spans V. Thus we can form a basis from B^* by deleting every vector x that is a linear combination of the vectors preceding x. Since the v_i's are independent of one another, none of the v_i's will be deleted during this procedure. Thus S is a subset of the resulting basis.

4. Let S be the set of all functions $f\colon \mathbb{R} \to \mathbb{R}$ that satisfy the differential equation

   $$y'' - y' + 2y = 0.$$

   Is S a real vector space? (Assume the usual operations $(f + g)(t) = f(t) + g(t)$ and $(c \cdot f)(t) = cf(t)$ where $c \in \mathbb{R}$.) Explain why or why not.

Solution. S is a vector space. Since the derivative is a linear transformation it is easily verified that S is closed under addition and scalar multiplication. The other vector space properties are easily seen as well.

5. Let A and B be $n \times n$ matrices. Show that AB is invertible if and only if A and B are invertible.

   Solution. ($\Longrightarrow$) Assume that AB is invertible. Suppose that B is not invertible. Thus there is a non-trivial solution to the homogenous system $B\mathbf{x} = \mathbf{0}$. This non-trivial solution will also solve the system $AB\mathbf{x} = \mathbf{0}$ contradicting the hypothesis that AB is invertible. So B must be invertible.

   Now given that AB and B are invertible, we have that $A = (AB)B^{-1}$ is the product of invertible matrices, whence A is invertible.

   ($\Longleftarrow$) If A and B are invertible, then AB is the product of invertible matrices and has inverse $B^{-1}A^{-1}$.

6. Prove that a linear transformation of vector spaces $L\colon V \to W$ is one to one if and only if L maps linearly independent subsets of V to linearly independent subsets of W.

   Solution. If V is the zero vector space the result is vacuously true. So assume that $V \neq \{0\}$.

   ($\Longrightarrow$) Assume that L is one to one. Let $\{v_1, \ldots, v_n\}$ be a linearly independent set in V. Suppose that there are constants $\{c_1, \ldots, c_n\}$ such that

   $$c_1 L v_1 + c_2 L v_2 + \cdots + c_n L v_n = 0$$

   Since L is linear we would then have

   $$L(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n) = 0$$

   Since L is one to one, this implies that $c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$ whence all the c_i's must be zero because of the linear independence of $\{v_1, \ldots, v_n\}$.

   ($\Longleftarrow$) Assume that L preserves linear independence. Let $v \in V$ be a non-zero vector. Then the set $\{v\}$ is linearly independent, whence $\{Lv\}$ is linearly independent. Thus $Lv \neq 0$, so L is one to one.

7. Let V be a finite dimensional vector space and $L : V \to W$ be a linear transformation to vector space W. Prove that dim(kernel L)+ dim(image L) = dim V.

Solution. Let n =dim V and k =dim $\ker L$. Note that $k \leq n$. If $k = n$ then $\ker L = V$ and image $L = \{0\}$ so the result is true.

So suppose that $0 \leq k < n$. Let $\{v_1, v_2, \ldots, v_k\} \subseteq V$ be a basis for $\ker L$. Since $\{v_1, v_2, \ldots, v_k\}$ is linearly independent, it can be extended to a basis

$$B_V = \{v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n\}$$

of V. We will show that $T = \{Lv_{k+1}, \ldots, Lv_n\}$ is a basis of the image of L.

To see that T spans let $w \in$ image L. Thus $w = Lv$ for some v in V. Write so there are constants $c_i (1 \leq i \leq n)$ such that $v = c_1v_1 + c_2v_2 + \cdots c_kv_k + c_{k+1}v_{k+1} + \cdots + c_nv_n$. Thus

$$w = Lv = 0 + c_{k+1}Lv_{k+1} + \cdots + c_nLv_n$$

whence T spans image L.

To see that T is linearly independent, suppose that there are constants $c_i (k + 1 \leq i \leq n)$ such that $c_{k+1}Lv_{k+1} + \cdots + c_nLv_n = 0$. Thus $c_{k+1}v_{k+1} + \cdots + c_nv_n$ is in the kernel of L whence there are constants $t_i (1 \leq i \leq k)$ such that $c_{k+1}v_{k+1} + \cdots + c_nv_n = t_1v_1 + \cdots + t_kv_k$. The fact that all the v_i's are independent then implies that all the c_i's (and all the t_i's) are zero.

Thus we've shown that dim(image L) $= n - k$ and the result follows.

8. Let A be an $n \times n$ matrix. Prove that A is invertible if and only if the determinant of A is non-zero.

Solution. ($\Longrightarrow$) If A is invertible then A is the product of elementary matrices. Since the determinant respect matrix multiplication and no elementary matrix has zero determinant, it follows that A has non-zero determinant.

($\Longleftarrow$) If A is not invertible then A is row equivalent to a matrix $\hat{A}$ that contains a row of zeros. Hence $A = E_1E_2 \cdots E_n\hat{A}$ where $E_1, E_2, \ldots, E_n$ is a sequence of elementary matrices. Thus

$$\det A = \det [E_1E_2 \cdots E_n\hat{A}] = \det (E_1) \det (E_2) \cdots \det (E_n) \det (\hat{A}) = 0$$

since $\det \hat{A} = 0$.