Mathematics Department Qualifying Exam Fall 2006
Subject: Algebra

Part A. Do **five** of the following eight problems.

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1. Let R be the ring of all continuous real valued functions on the closed interval $[0, 1]$. Prove that the map $\phi : R \to \mathbb{R}$ defined by $\phi(f) = \int_0^1 f(t)\, dt$ is a homomorphism of additive groups but not a ring homomorphism.

2. Show that the symmetric group S_n $(n \geq 2)$ is generated by the 2-cycles $(1\ 2), (2\ 3), \ldots, (n-1\ n)$.

3. Let $\mathbb{R}^\times$ denote the multiplicative group of nonzero real numbers and $\mathbb{R}$ denote the additive group of real numbers. Show that $\mathbb{R}^\times \cong \mathbb{R} \times \mathbb{Z}_2$.

4. An element x in a ring R is called *nilpotent* if $x^m = 0$ for some $m \in \mathbb{Z}^+$. Let R be a commutative ring with $1 \neq 0$. Prove that if a is a nilpotent element of R, then $1 - ab$ is a unit for all $b \in R$.

5. (a) Let $H = \{(1), (2\ 3)\}$. Is H normal in S_3?

   (b) What is the order of the element $14 + \langle 8 \rangle$ in the quotient group $\mathbb{Z}_{24}/\langle 8 \rangle$?

6. Prove that any subfield of $\mathbb{R}$ must contain $\mathbb{Q}$.

7. Prove that if H and K are finite subgroups of G whose orders are relatively prime then $H \cap K = 1$.

8. Consider the additive quotient group $\mathbb{Q}/\mathbb{Z}$.

   (a) Show that every coset of $\mathbb{Z}$ in $\mathbb{Q}$ contains exactly one representative $q \in \mathbb{Q}$ in the range $0 \leq q < 1$.

   (b) Show that every element of $\mathbb{Q}/\mathbb{Z}$ has finite order but that there are elements of arbitrarily large order.

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Part B. Do **five** of the following eight problems.

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1. Determine the dimension of the solution space (over the real numbers) to the system of equations

$$\begin{array}{rcrcrcrcrcl} x_1 & + & x_2 & & & & & & & = & 0 \\ & & x_2 & + & x_3 & & & & & = & 0 \\ & & & & x_3 & + & x_4 & & & = & 0 \\ & & & & & & x_4 & + & x_5 & = & 0 \\ -x_1 & & & & & & & + & x_5 & = & 0 \end{array}$$

2. Find a *unit* vector in $\mathbf{R}^3$ that is mutually perpendicular to $v = (1, 2, 3)$ and $w = (3, 2, 1)$.

3. Suppose

$$A = \begin{pmatrix} 1 & -1 & 0 & 2 \\ -2 & 0 & 0 & 1 \\ -1 & -1 & 0 & 3 \end{pmatrix}.$$

Determine a basis over the reals for the image of the linear transformation $L(\mathbf{v}) = A\mathbf{v}$.

4. Determine the inverse of the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{pmatrix}$$

5. Suppose A is an invertible matrix. Prove $(A^t)^{-1} = (A^{-1})^t$ where M^t and M^{-1} represent the transpose and inverse, respectively, of the matrix M.

6. Determine an *orthonormal basis* for $\mathbf{span}\{(-1, 0, 0, 1), (0, 0, 1, -1), (1, -1, 0, 0)\}$.

7. State what it means (the definition) for a finite set of vectors to be linearly independent, and use this definition to prove the set of vectors $\{(1, 1, 0), (0, 1, 1), (1, 0, 1)\}$ is linearly independent.

8. Determine the eigenvalues of the matrix

$$\begin{pmatrix} 2 & 2 & -5 \\ 3 & 7 & -15 \\ 1 & 2 & -4 \end{pmatrix}$$