Algebra Qualifying Exam, Solutions Fall 2006

Part A.

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1. Let R be the ring of all continuous real valued functions on the closed interval $[0,1]$. Prove that the map $\phi : R \to \mathbb{R}$ defined by $\phi(f) = \int_0^1 f(t)\ dt$ is a homomorphism of additive groups but not a ring homomorphism.

   Solution. We know that

   $$\int (f+g)\ dt = \int f\ dt + \int g\ dt$$

   using this it is easy to show that ϕ is a homomorphism of additive groups. In fact

   $$\phi(f+g) = \int_0^1 [f(t)+g(t)]\ dt = \int_0^1 f(t)\ dt + \int_0^1 g(t)\ dt = \phi(f)+\phi(g)$$

   However, for the functions $f(t) = t$ and $g(t) = t$ we get

   $$\phi(fg) = \int_0^1 t^2\ dt = \frac{1}{3} \neq \frac{1}{4} = \left(\int_0^1 t\ dt\right)^2 = \phi(f)\phi(g)$$

2. Show that the symmetric group S_n $(n \geq 2)$ is generated by the 2-cycles $(1\ 2), (2\ 3), \ldots, (n-1\ n)$.

   Solution. First let us show that using these 2-cycles we can construct any 2-cycle. Call the set of elements above T.

   Let $(a\ b)$ be a 2-cycle in S_n. WLOG assume $b = a+k+1$ for some positive integer k.

   Note that $(a\ a+1)$ is one of the cycles we can use, and that

   $$(a+1\ b) = (a\ a+1)(a\ b)(a\ a+1)$$

   We now repeat this with $(a\ a+2)$ to get

   $$(a{+}2\ b) = (a{+}1\ a{+}2)(a{+}1\ b)(a{+}1\ a{+}2) = (a{+}1\ a{+}2)(a\ a{+}1)(a\ b)(a\ a{+}1)(a{+}1\ a{+}2)$$

   ... etcetera, until we get

   $$(a{+}k\ \ b) = (a{+}k{-}1\ \ a{+}k)\cdots(a\ a{+}2)(a\ a{+}1)(a\ b)(a\ a{+}1)(a{+}1\ a{+}2)\cdots(a{+}k{-}1\ \ a{+}k)$$

   Since $b = a+k+1$, then the left hand side is in T, and the right hand side is a product of elements in T and (ab). If we move most of the things to the left hand side (multiplying by inverses) we get

   $$(a\ a{+}1)(a{+}1\ a{+}2)\cdots(a{+}k{-}1\ \ a{+}k)(a{+}k\ \ b)(a{+}k{-}1\ \ a{+}k)\cdots(a{+}1\ a{+}2)(a\ a{+}1) = (a\ b)$$

   which means that $(a\ b)$ is generated by elements in T.

   Since every element in S_n can be written as a product of 2-cycles, then T generates all S_n.

3. Let $\mathbb{R}^\times$ denote the multiplicative group of nonzero real numbers and $\mathbb{R}$ denote the additive group of real numbers. Show that $\mathbb{R}^\times \cong \mathbb{R} \times \mathbb{Z}_2$.

   Solution. Think $\mathbb{Z}_2$ as the multiplicative group $\{1, -1\}$. Define a function $sign(x) : \mathbb{R}^\times \to \mathbb{Z}_2$ by $sign(x) = 1$ if x is positive, and $sign(x) = -1$ if x is negative.

   Note that $sign(xy) = sign(x)sign(y)$ for all non-zero real numbers x and y.

   Consider the map $\phi : \mathbb{R}^\times \to \mathbb{R} \times \mathbb{Z}_2$ defined by $\phi(x) = (\ln|x|, sign(x))$ (I know it is ugly-defined as I am using additive notation for the first component and multiplicative for the second).

   Now note that

   $$\phi(xy) = (\ln|xy|, sign(xy)) = (\ln|x|+\ln|y|, sign(x)sign(y)) = (\ln|x|, sign(x))(\ln|y|, sign(y))$$

   So, ϕ is a homomorphism. It is easy to see that ϕ is onto using that $\ln|x|$ is onto $\mathbb{R}$. Moreover,

   $$Ker(\phi) = \{x \in \mathbb{R}^\times;\ \ln|x| = 0 \text{ and } sign(x) = 1\} = \{1\}$$

   Hence, ϕ defines an isomorphism.

4. An element x in a ring R is called *nilpotent* if $x^m = 0$ for some $m \in \mathbb{Z}^+$. Let R be a commutative ring with $1 \neq 0$. Prove that if a is a nilpotent element of R, then $1 - ab$ is a unit for all $b \in R$.

   Solution. First notice that if $a^m = 0$ for some m, then $(ab)^m = 0$ for the same m (here using the ring is commutative). So, let's call $ab = x$, we know x is nilpotent and that the m^{th} power kills it.

   Now consider the product

   $$(1 - x)(1 + x + x^2 + \cdots + x^{m-1}) = 1 - x^m = 1$$

   It follows that the inverse of $1 - x$ is $1 + x + x^2 + \cdots + x^{m-1}$ (whatever that is, anyway it will be an element of the ring).

5. (a) Let $H = \{(1), (2\ 3)\}$. Is H normal in S_3?

   (b) What is the order of the element $14 + \langle 8 \rangle$ in the quotient group $\mathbb{Z}_{24}/\langle 8 \rangle$?

   Solution.

   (a) No, as $(123)(23)(123)^{-1} = (13) \notin H$

   (b) The group $H =< 8 >$ has the elements $H = \{0, 8, 16\}$.
   Now I will compute the 'powers' of 14 all modulo 24
   $14 \notin H$
   $14 + 14 = 28 \equiv 4 \notin H$
   $14 + 14 + 14 \equiv 4 + 14 \equiv 18 \notin H$
   $14 + 14 + 14 + 14 \equiv 4 + 4 \equiv 8 \in H$
   Hence, the order of $14+ < 8 >$ is 4

6. Prove that any subfield of $\mathbb{R}$ must contain $\mathbb{Q}$.

Solution. A field always F has a one. By closure of the addition of F we can see that all the integers must be in F (here we are using that $\mathbb{Z} \subset \mathbb{R}$). Since we need inverses for all non-zero elements in F, then all the elements of the form $\frac{1}{x}$, for $x \in \mathbb{Z}$ must be in F as well. Finally, using closure of the multiplication we can construct any rational as the product of two elements in F,

$$\frac{a}{b} = a \cdot \frac{1}{b}$$

and thus $\mathbb{Q} \subset F$.

7. Prove that if H and K are finite subgroups of G whose orders are relatively prime then $H \cap K = 1$.

Solution. This is problem 4 part A in the exam of Fall 2007

8. Consider the additive quotient group $\mathbb{Q}/\mathbb{Z}$.

   (a) Show that every coset of $\mathbb{Z}$ in $\mathbb{Q}$ contains exactly one representative $q \in \mathbb{Q}$ in the range $0 \leq q < 1$.

   (b) Show that every element of $\mathbb{Q}/\mathbb{Z}$ has finite order but that there are elements of arbitrarily large order.

Solution.

(a) Let $x \in \mathbb{Q}$, we denote the greatest integer that is less or equal to x as $[x]$, then $x - [x] \in [0, 1)$. Since $[x]$ is an integer, then $x + \mathbb{Z} = x - [x] + \mathbb{Z}$. So, every coset of $\mathbb{Z}$ in $\mathbb{Q}$ contains at least one representative in $[0, 1)$. Moreover, having two representatives in $[0, 1)$ will force a number x to have two decimal parts, that is not possible.

(b) Let $x = \frac{a}{b} \in \mathbb{Q}$. WLOG we take $b > 0$. Then, the order of $x + \mathbb{Z}$ is at most b, as

$$\frac{a}{b} + \cdots + \frac{a}{b} \quad (b \text{ times})$$

is equal to $a \in \mathbb{Z}$

So, every element in $\mathbb{Q}/\mathbb{Z}$ has finite order.

Now consider a number N (as large as you wish), we know there are infinitely many primes, thus there must be a prime number p that is larger than N. It is easy to see that $x = \frac{1}{p}$ has order $p > N$.

Part B.

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1. Determine the dimension of the solution space (over the real numbers) to the system of equations

$$\begin{array}{rcrcrcrcrcl} x_1 & + & x_2 & & & & & & & = & 0 \\ & & x_2 & + & x_3 & & & & & = & 0 \\ & & & & x_3 & + & x_4 & & & = & 0 \\ & & & & & & x_4 & + & x_5 & = & 0 \\ -x_1 & & & & & & & + & x_5 & = & 0 \end{array}$$

Solution. The last equation in the system says $x_5 = x_1$, the previous to the last says $x_4 = -x_5$, which together with the last one imply $x_1 = -x_4$. Since $x_3 = -x_4$ we get that $x_1 = x_3$.

The solution space of this system is spanned by $(1, -1, 1, -1, 1)$. So, the dimension is one.

2. Find a *unit* vector in $\mathbf{R}^3$ that is mutually perpendicular to $v = (1, 2, 3)$ and $w = (3, 2, 1)$.

Solution. We will first find a vector that is perpendicular to v and w, then we will normalize it.

The vector we are looking for is $u = (x, y, z)$, it follows that

$$0 = (x, y, z) \cdot (1, 2, 3) = x + 2y + 3z \qquad \text{and} \qquad 0 = (x, y, z) \cdot (3, 2, 1) = 3x + 2y + z$$

So, subtracting these two equations we get $2x - 2z = 0$, which implies $x = z$. Plugging this into one of the equations we get $4x + 2y = 0$, which implies $y = -2x$. Hence, the solution space for the system of equations above is spanned by $(1, -2, 1)$.

So, any vector that is a multiple of $(1, -2, 1)$ is orthogonal to both v and w. In particular, we take

$$\frac{1}{|(1, -2, 1)|}(1, -2, 1) = \frac{1}{\sqrt{6}}(1, -2, 1)$$

which has norm one.

3. Suppose

$$A = \begin{pmatrix} 1 & -1 & 0 & 2 \\ -2 & 0 & 0 & 1 \\ -1 & -1 & 0 & 3 \end{pmatrix}.$$

Determine a basis over the reals for the image of the linear transformation $L(\mathbf{v}) = A\mathbf{v}$.

Solution. We know the image of L is spanned by the column vectors of the matrix A. Since one of the column vectors is the zero vector, then we just have to play with

$$B = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 0 & 1 \\ -1 & -1 & 3 \end{pmatrix}.$$

(which has determinant zero, thus we need to get rid of columns)

We will do some column operations on this matrix to find a basis for $Im(L)$

$$\begin{aligned} B &= \begin{pmatrix} 1 & -1 & 2 \\ -2 & 0 & 1 \\ -1 & -1 & 3 \end{pmatrix} && \text{now we add column 2 to column 1} \\ &= \begin{pmatrix} 0 & -1 & 2 \\ -2 & 0 & 1 \\ -2 & -1 & 3 \end{pmatrix} && \text{now we add column 2 twice to column 3} \\ &= \begin{pmatrix} 0 & -1 & 0 \\ -2 & 0 & 1 \\ -2 & -1 & 1 \end{pmatrix} \end{aligned}$$

Now we can see clearly that column 1 and column 3 are linearly dependent. So, a basis for $Im(L)$ is given by

$$\{(-1, 0, -1), (0, 1, 1)\}$$

4. Determine the inverse of the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{pmatrix}$$

Solution. Since this is an anti-diagonal matrix then its inverse is also anti-diagonal and given by

$$\begin{pmatrix} 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

5. Suppose A is an invertible matrix. Prove $(A^t)^{-1} = (A^{-1})^t$ where M^t and M^{-1} represent the transpose and inverse, respectively, of the matrix M.

Solution. We know that $(AB)^t = B^t A^t$, then

$$I = (A^{-1}A)^t = A^t \left(A^{-1}\right)^t$$

and

$$I = (AA^{-1})^t = \left(A^{-1}\right)^t A^t$$

6. Determine an *orthonormal basis* for $\mathbf{span}\{(-1, 0, 0, 1), (0, 0, 1, -1), (1, -1, 0, 0)\}$.

Solution. We will use the Gram-Schmidt process.

Note that the vectors $v_1 = (0, 0, 1, -1)$ and $v_2 = (1, -1, 0, 0)$ are already orthogonal. So, we just need to find the third vector

Then

$$\begin{aligned} v_3 &= (-1,0,0,1) - \frac{(0,0,1,-1)\cdot(-1,0,0,1)}{|(0,0,1,-1)|^2}(0,0,1,-1) - \frac{(1,-1,0,0)\cdot(-1,0,0,1)}{|(1,-1,0,0)|^2}(1,-1,0,0) \\ &= (-1,0,0,1) + \frac{1}{2}(0,0,1,-1) + \frac{1}{2}(1,-1,0,0) \\ &= \left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \end{aligned}$$

So, v_1, v_2 and v_3 are orthogonal and span the same subspace the original three vectors spanned. Now we normalize these vectors by dividing them by their norm. The final answer is

$$\left\{\frac{1}{\sqrt{2}}(0,0,1,-1), \frac{1}{\sqrt{2}}(1,-1,0,0), \left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)\right\}$$

7. State what it means (the definition) for a finite set of vectors to be linearly independent, and use this definition to prove the set of vectors $\{(1,1,0),(0,1,1),(1,0,1)\}$ is linearly independent.

Solution. A set of vectors is linearly independent if the only linear combination of these vectors that yields the zero vector is the trivial one (i.e. all scalars must be zero).

Assume that

$$(0,0,0) = \alpha(1,1,0) + \beta(0,1,1) + \gamma(1,0,1) = (\alpha+\gamma, \alpha+\beta, \beta+\gamma)$$

then we get the system

$$\alpha + \gamma = 0 \qquad \alpha + \beta = 0 \qquad \beta + \gamma = 0$$

which has only one solution $\alpha = \beta = \gamma = 0$.

8. Determine the eigenvalues of the matrix

$$\begin{pmatrix} 2 & 2 & -5 \\ 3 & 7 & -15 \\ 1 & 2 & -4 \end{pmatrix}$$

Solution. Let's play with this determinant

$$\begin{aligned}
\chi_A(\lambda) &= \begin{vmatrix} 2-\lambda & 2 & -5 \\ 3 & 7-\lambda & -15 \\ 1 & 2 & -4-\lambda \end{vmatrix} && \text{now we multiply row 1 by 3} \\
&= \frac{1}{3}\begin{vmatrix} 6-3\lambda & 6 & -15 \\ 3 & 7-\lambda & -15 \\ 1 & 2 & -4-\lambda \end{vmatrix} && \text{now we subtract row 2 from row 1} \\
&= \frac{1}{3}\begin{vmatrix} 3-3\lambda & -1+\lambda & 0 \\ 3 & 7-\lambda & -15 \\ 1 & 2 & -4-\lambda \end{vmatrix} && \text{now we multiply column 2 by 3} \\
&= \frac{1}{3^2}\begin{vmatrix} 3-3\lambda & -3+3\lambda & 0 \\ 3 & 21-3\lambda & -15 \\ 1 & 6 & -4-\lambda \end{vmatrix} && \text{now we add column 1 to column 2} \\
&= \frac{1}{3^2}\begin{vmatrix} 3-3\lambda & 0 & 0 \\ 3 & 24-3\lambda & -15 \\ 1 & 7 & -4-\lambda \end{vmatrix} \\
&= \frac{1}{3^2}(3-3\lambda)\begin{vmatrix} 24-3\lambda & -15 \\ 7 & -4-\lambda \end{vmatrix} \\
&= (1-\lambda)\begin{vmatrix} 8-\lambda & -5 \\ 7 & -4-\lambda \end{vmatrix} \\
&= (1-\lambda)[(8-\lambda)(-4-\lambda)+35] \\
&= (1-\lambda)(\lambda^2-4\lambda)+3) \\
&= (1-\lambda)(\lambda-3)(\lambda-1)
\end{aligned}$$

So, the eigenvalues are $\lambda = 1$ (with multiplicity 2) and $\lambda = 3$ (with multiplicity one).