Mathematics Department Qualifying Exam : Algebra Fall 2007

Part A. Solve **five** of the following eight problems :

1. TRUE/FALSE : Let G be a group and $a, b \in G$. If ab has order 3 then ba has order 3.
Prove your answer!

2. TRUE/FALSE : Let R be an integral domain and A a proper ideal of R. Then R/A is an integral domain.
Prove your answer!

3. TRUE/FALSE : $D_{12} \cong \mathbb{Z}_3 \oplus D_4$
Prove your answer!

4. Let G be a group and H and K finite subgroups of G such that $|H|$ and $|K|$ are relatively prime. Prove that $H \cap K = \{1\}$.

5. Let G be a group. Consider the map $f : G \to G : a \to a^{-1}$. Prove that G is abelian if and only if f is a group homomorphism.

6. Let $\mathbb{R}^*$ be the group of nonzero real numbers under multiplication and

$$H = \{g \in \mathbb{R}^* \mid g^m \in \mathbb{Q} \text{ for some nonzero integer } m\}$$

Prove that H is a subgroup of $\mathbb{R}^*$.

7. Find an element of order 10 in A_9. Prove that the order is indeed 10.

8. Let R be the set of 2×2-matrices with real entries :

$$R = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

Then R forms a ring under matrix addition and matrix multiplication. Put

$$S = \left\{ \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \in R \right\}$$

- Prove that S is a subring of R.
- Is S an ideal of R?

Part B is on the back!!!

Part B. Solve **five** of the following eight problems :

1. Let A be an $n \times n$-matrix. Prove that $\det(AA^T) \geq 0$.

2. Let $\mathbf{u}$ be a fixed vector in $\mathbb{R}^n$. Show that the set of all vectors in $\mathbb{R}^n$ that are orthogonal to $\mathbf{u}$ is a subspace of $\mathbb{R}^n$.

3. Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$. Find a matrix P such that $P^{-1}AP$ is a diagonal matrix.

4. Let A be an $n \times n$-matrix and λ and eigenvalue of A. Prove that λ^k is an eigenvalue of A^k for all positive integers k.

5. Let $L : V \to W$ be a linear transformation. If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ spans V, show that $\{L(\mathbf{v}_1), L(\mathbf{v}_2), \ldots, L(\mathbf{v}_k)\}$ spans range(L).

6. Find an orthogonal basis for

$$S = \text{span}\left\{ \begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix}^T, \begin{bmatrix} 0 & 1 & -1 & 1 \end{bmatrix}^T, \begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix}^T \right\}$$

7. Let A and B be symmetric $n \times n$-matrices. Prove that AB is symmetric if and only if $AB = BA$.

8. Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of linearly independent vectors in $\mathbb{R}^n$. Prove that

$$\{\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3, \ldots, \mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_k\}$$

is also linearly independent.