Algebra Qualifying Exam, Solutions Fall 2007

Part A.

1. TRUE/FALSE : Let G be a group and $a, b \in G$. If ab has order 3 then ba has order 3.

Solution. Note that

$$(ba)^3 = b(ab)^2a = b(ab)^{-1}a = b(b^{-1}a^{-1})a = e$$

Then, the order of ba is one or three. If it were one, then $b = a^{-1}$, which contradicts ab having order 3.

2. TRUE/FALSE : Let R be an integral domain and A a proper ideal of R. Then R/A is an integral domain.

Solution. False. For n not a prime number, the ring $\mathbb{Z}/n\mathbb{Z}$ is not an integral domain.

3. TRUE/FALSE : $D_{12} \cong \mathbb{Z}_3 \oplus D_4$

Solution. False. D_{12} has 13 elements of order 2, but since D_4 has exactly 5 elements of order 2 and $\mathbb{Z}_3$ has none, then $\mathbb{Z}_3 \oplus D_4$ has just 5 elements of order 2.

4. Let G be a group and H and K finite subgroups of G such that $|H|$ and $|K|$ are relatively prime. Prove that $H \cap K = \{1\}$.

Solution. We know that the intersection of two subgroups, H and K, is also a subgroup (that is a subgroup of both H and K). So, the order of $H \cap K$ divides both $|H|$ and $|K|$, thus $|H \cap K|$ must be one, as $(|H|, |K|) = 1$.

5. Let G be a group. Consider the map $f : G \to G : a \to a^{-1}$. Prove that G is abelian if and only if f is a group homomorphism.

Solution. We know that $(ab)^{-1} = b^{-1}a^{-1}$.
Assuming f is a homomorphism

$$b^{-1}a^{-1} = (ab)^{-1} = f(ab) = f(a)f(b) = a^{-1}b^{-1}$$

Since every element in a group has an inverse, then $cd = dc$ for all $c, d \in G$

Assuming that G is Abelian.

$$f(ab) = (ab)^{-1} = b^{-1}a^{-1} = a^{-1}b^{-1} = f(a)f(b)$$

6. Let $\mathbb{R}^*$ be the group of nonzero real numbers under multiplication and

$$H = \{g \in \mathbb{R}^* \ | g^m \in \mathbb{Q} \text{ for some nonzero integer } m\}$$

Prove that H is a subgroup of $\mathbb{R}^*$.

Solution. Let $g, h \in H$, then there are integers n, m such that $g^n \in \mathbb{Q}$ and $h^m \in \mathbb{Q}$. Now consider

$$(gh^{-1})^{mn} = g^{mn}(h^{-1})^{mn} = (g^n)^m(h^m)^{-n}$$

which is in $\mathbb{Q}$ because $\mathbb{Q}^*$ is a multiplicative group.
Since $1 \in H$, then H is a subgroup of $\mathbb{R}^*$.

7. Find an element of order 10 in A_9. Prove that the order is indeed 10.

Solution. Consider $\sigma = (12)(34)(56789)$.
Since

$$\sigma = (12)(34567) = (12)(34)(59)(58)(57)(56)$$

then $\sigma \in A_9$.
The order of σ is 10 because the order of a product of disjoint cycles is the lcm of the orders of the cycles. In our case, the order of σ is $lcm(2, 5) = 10$.

8. Let R be the set of 2×2-matrices with real entries :

$$R = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

Then R forms a ring under matrix addition and matrix multiplication. Put

$$S = \left\{ \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \in R \right\}$$

- Prove that S is a subring of R.
- Is S an ideal of R?

Solution. This is problem number 4 in part A in the exam of Spring 2007.

Part B.

1. Let A be an $n \times n$-matrix with entries in $\mathbb{R}$. Prove that $\det(AA^T) \geq 0$.

Solution. We know that $\det(A) = \det(A^T)$, and that $\det(AB) = \det(A)\det(B)$. So,

$$\det(AA^T) = \det(A)\det(A^T) = \det(A)^2$$

Since $\det(A) \in \mathbb{R}$, then $\det(AA^T) = \det(A)^2 \geq 0$.

2. Let $\mathbf{u}$ be a fixed vector in $\mathbb{R}^n$. Show that the set of all vectors in $\mathbb{R}^n$ that are orthogonal to $\mathbf{u}$ is a subspace of $\mathbb{R}^n$.

Solution. Let $\mathbf{v}$ and $\mathbf{w}$ be two vectors that are orthogonal with $\mathbf{u}$, and let $\alpha \in \mathbb{R}$. Then,

$$\mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{u} = 0 - 0 = 0$$

and

$$\mathbf{u} \cdot (\alpha \mathbf{v}) = \alpha(\mathbf{u} \cdot \mathbf{v}) = \alpha(0) = 0$$

Since the zero vector is orthogonal to $\mathbf{u}$, then the set of orthogonal vectors to $\mathbf{u}$ is non-empty.

3. Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$. Find a matrix P such that $P^{-1}AP$ is a diagonal matrix.

Solution. We first look at the characteristic polynomial of A

$$\begin{aligned} \chi_A(\lambda) &= \begin{vmatrix} 1-\lambda & 3 \\ 2 & 2-\lambda \end{vmatrix} \\ &= (1-\lambda)(2-\lambda) - 6 \\ &= \lambda^2 - 3\lambda - 4 \\ &= (\lambda - 4)(\lambda + 1) \end{aligned}$$

We know that there is a matrix P such that

$$P^{-1}AP = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$$

In order to find P we need to find eigenvectors for the two eigenvalues of A.
For $\lambda = -1$ we have to solve the equation $A\mathbf{v} = -\mathbf{v}$, which yields the system of equations

$$x + 3y = -x \qquad\qquad 2x + 2y = -y$$

which has solution space spanned by $(3, -2)$.
For $\lambda = 4$ we have to solve the equation $A\mathbf{v} = 4\mathbf{v}$, which yields the system of equations

$$x + 3y = 4x \qquad\qquad 2x + 2y = 4y$$

which has solution space spanned by $(1, 1)$.
It follows that

$$P = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$$

4. Let A be an $n \times n$-matrix and λ and eigenvalue of A. Prove that λ^k is an eigenvalue of A^k for all positive integers k.

Solution. Let $\mathbf{v}$ be an eigenvector of A associated to the eigenvalue λ, that is $A\mathbf{v} = \lambda\mathbf{v}$.
Now note that

$$A^k\mathbf{v} = A^{k-1}(A\mathbf{v}) = A^{k-1}(\lambda\mathbf{v}) = \lambda(A^{k-1}\mathbf{v})$$

Repeating the process above we see that $A^k\mathbf{v} = \lambda^k\mathbf{v}$.

5. Let $L : V \to W$ be a linear transformation. If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ spans V, show that $\{L(\mathbf{v}_1), L(\mathbf{v}_2), \ldots, L(\mathbf{v}_k)\}$ spans range(L).

Solution. Let $\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k \in V$. Then

$$L(\mathbf{v}) = L(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k) = \alpha_1 L(\mathbf{v}_1) + \alpha_2 L(\mathbf{v}_2) + \cdots + \alpha_k L(\mathbf{v}_k)$$

So, every element in the range of L is a linear combination of the elements in the set $\{L(\mathbf{v}_1), L(\mathbf{v}_2), \ldots, L(\mathbf{v}_k)\}$.

6. Find an orthogonal basis for

$$S = \text{span}\left\{ \begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix}^T, \begin{bmatrix} 0 & 1 & -1 & 1 \end{bmatrix}^T, \begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix}^T \right\}$$

Solution. Note that $\mathbf{v}_2 = \begin{bmatrix} 0 & 1 & -1 & 1 \end{bmatrix}^T$ and $\mathbf{v}_3 = \begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix}^T$ are already orthogonal. So, what we want to do is to replace $\mathbf{v}_1 = \begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix}^T$ with a vector in S that is orthogonal to the last two.
Any element in S looks like $\mathbf{v} = \begin{bmatrix} x+z, & x+y, & z-y, & x+y+z \end{bmatrix}^T$ for some $x, y, z \in \mathbb{R}$.
Note that

$$\mathbf{v} \cdot \mathbf{v}_2 = x + y + z - y + x + y + z = 2x + y + 2z$$

and

$$\mathbf{v} \cdot \mathbf{v}_3 = x + z + z - y + x + y + z = 2x + 3z$$

Since we want $\mathbf{v}$ to be orthogonal to both $\mathbf{v}_2$ and $\mathbf{v}_3$, then

$$2x + y + 2z = 0 = 2x + 3z$$

which forces $y = z$, and thus we get $0 = 2x + 3z$. We now may take $x = 3$ and then $y = z = -2$. Finally,

$$\mathbf{v} = \begin{bmatrix} 3-2, & 3-2, & 0, & 3-2-2 \end{bmatrix}^T = \begin{bmatrix} 1 & 1 & 0 & -1 \end{bmatrix}^T$$

So,

$$\left\{ \begin{bmatrix} 1 & 1 & 0 & -1 \end{bmatrix}^T, \begin{bmatrix} 0 & 1 & -1 & 1 \end{bmatrix}^T, \begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix}^T \right\}$$

is an orthogonal basis for S.
Of course you may solve this using projections, but I thought this way was much more fun.

7. Let A and B be symmetric $n \times n$-matrices. Prove that AB is symmetric if and only if $AB = BA$.

Solution. Assuming $AB = BA$, $A^T = A$ and $B^T = B$.

$$(AB)^T = B^T A^T = BA = AB$$

So, AB is symmetric.
Now assuming that AB is symmetric, $A^T = A$ and $B^T = B$.

$$AB = (AB)^T = B^T A^T = BA$$

8. Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of linearly independent vectors in $\mathbb{R}^n$. Prove that

$$\{\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3, \ldots, \mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_k\}$$

is also linearly independent.

Solution. This is problem 1 in part B in the exam of Spring 2007