
Part A. Do **five** of the following eight problems :

1. Show there are no integers n such that
   (a) $S_2 \times S_5$ is isomorphic to S_n.
   (b) $S_3 \times \mathbb{Z}/\mathbb{Z}_4$ is isomorphic to S_n.

2. Let G be a cyclic group of order 8. Prove that G has exactly one element of order 2. Does there exist a nonabelian group of order 8 with this property?

3. Prove that the map $f : G \to G$, given by $f(a) = a^{-1}$, is a group homomorphism if and only if the map $g : G \to G$, given by $g(a) = a^2$, is a group homomorphism.

4. What is the order of $\sigma = (1\ 10\ 4)(2\ 13)(1\ 12\ 8)(5\ 7)(6\ 9)(5\ 11)$?

5. Let
$$G = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \ ;\ a, b, c \in \mathbb{R},\ ac \neq 0 \right\}$$
Show that G is a group under matrix multiplication, and that
$$H = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \ ;\ b \in \mathbb{R} \right\}$$
is a subgroup of G.

6. Let G be a group and H a subgroup of G. Let N_H be the set of all $x \in G$ such that $xHx^{-1} = H$. Show that N_H is a subgroup of G containing H, and that H is normal in N_H. (The group N_H is called *the normalizer of* H.)

7. Define a relation $\sim$ on the set $\mathbb{N}$ of natural numbers by $a \sim b$ if and only if $a^2 + b^2$ is even. Prove that $\sim$ is an equivalence relation.

8. Let F be a field and R any ring, and let $\varphi\colon F \to R$ be a nonzero ring homomorphism. Prove that $\ker(\varphi) = 0$.

Part B. Solve **five** of the following eight problems :

1. Let v_1, v_2, v_3 be linearly independent vectors in a vector space V. Show that the vectors $v_1, 2v_1 + v_2, 3v_1 + 2v_2 + v_3$ are also linearly independent.

2. Find explicitly a linear map $f : \mathbb{R}^2 \to \mathbb{R}^3$ such that $f(2,3) = (1,2,3)$ and $f(1,2) = (4,5,6)$. Is this linear function unique?

3. Give examples of three of the following:

   (a) Three vectors in $\mathbb{R}^3$ so that any two are linear independent, but the set of all three is linearly dependent.

   (b) Matrices M and N such that $MN \neq NM$.

   (c) Linear maps
   $$f : \mathbb{R}^2 \to \mathbb{R}^4 \qquad\qquad g : \mathbb{R}^4 \to \mathbb{R}^2$$
   such that $g \circ f$ is invertible.

   (d) A matrix M such that $M^3 = 0$ but $M^2 \neq 0$.

4. Let $\mathcal{P}_2 = \{p(x) = a + bx + cx^2;\ a, b, c \in \mathbb{R}\}$. Show that the map $T : \mathcal{P}_2 \to \mathcal{P}_2$ given by
   $$T(p) = \frac{d^2}{dx^2}\big((x^2+1)p(x)\big)$$
   is linear. Then find the matrix of T in the basis $\{1, x, x^2\}$.

5. Consider the following vector subspaces of $\mathbb{R}^3$
   $$A = \{(x,y,z) \in \mathbb{R}^3 |\ 2x+y+3z = 0\},\ B = \text{span}\{(1,-2,0)\},\ C = \text{span}\{(1,1,-1)\}.$$
   Show that A is the *direct sum* of B and C, that is $A = B \oplus C$.

6. Consider the subspace $U = \{(x,y,z) \in \mathbb{R}^3 | x + y - 2z = 0\} \subset \mathbb{R}^3$ and the system of vectors $S = \{v_1 = (1,1,1), v_2 = (5,-1,2)\} \subset \mathbb{R}^3$.

   (a) Show that U is the *span* of S.

   (b) Complete S to a basis in $\mathbb{R}^3$.

7. Let T be a linear operator on a vector space V, and let x be an eigenvector of T corresponding to the eigenvalue λ. For any positive integer m, prove that x is an eigenvector of T^m corresponding to the eigenvalue λ^m.

8. If an $n \times n$ matrix A is diagonalizable, show that $\det(A)$ is equal to the product of the eigenvalues of A.