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## Part A.

1. Show there are no integers $n$ such that

   (a) $S_2 \times S_5$ is isomorphic to $S_n$.

   (b) $S_3 \times \mathbb{Z}/\mathbb{Z}_4$ is isomorphic to $S_n$.

   **Solution:**

   (a) The order of $S_2 \times S_5$ is $2! \cdot 5! = 240$, which is not an $n!$

   (b) The order of $G = S_3 \times \mathbb{Z}/\mathbb{Z}_4$ is $3! \cdot 4 = 4!$, so the only case to check is whether or not $G$ is isomorphic to $S_4$. However, the element $((123), 1) \in G$ has order 12, but there are no elements of order 12 in $S_4$. So, the groups cannot be isomorphic.

2. Let $G$ be a cyclic group of order 8. Prove that $G$ has exactly one element of order 2. Does there exist a nonabelian group of order 8 with this property?

   **Solution:** A (multiplicative) cyclic group $G$ of order 8 can be written as

   $$G = \{e, g, g^2, g^3, g^4, g^5, g^6, g^7\}$$

   where $g^8 = e$.

   It is easy to see that the orders of the non-trivial elements are

   $$o(g) = 8 \qquad o(g^2) = 4 \qquad o(g^3) = 8 \qquad o(g^4) = 2$$
   $$o(g^5) = 8 \qquad o(g^6) = 4 \qquad o(g^7) = 8$$

   Thus, $g^4$ is the only element of order two in $G$.

   There are exactly two (non-isomorphic) non-abelian groups of order 8, namely $D_4$ and $Q_8$. If one thinks $D_4$ as the group of symmetries of a square, it is easy to see that the square admits 4 different reflections, these reflections have order two, so $D_4$ has at least 4 elements of order 2. On the other hand

   $$Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$$

   where $i^2 = j^2 = k^2 = -1$ (and some other multiplication rules that we don't need right now). It follows that $x^2 = -1$ for all $x \in Q_8 \setminus \{\pm 1\}$. Hence, $-1$ is the only element of order two in $Q_8$.

3. Prove that the map $f : G \to G$, given by $f(a) = a^{-1}$, is a group homomorphism if and only if the map $g : G \to G$, given by $g(a) = a^2$, is a group homomorphism.

**Solution:** The easiest way to prove this could be to show that the two statements we want to show are equivalent are also equivalent with $G$ being Abelian.

First of all, if $G$ is Abelian, then the two functions are clearly homomorphisms.

Now assume that $f$ is a group homomorphism, then

$$b^{-1}a^{-1} = (ab)^{-1} = f(ab) = f(a)f(b) = a^{-1}b^{-1}$$

Since every element in a group is the inverse of some element in $G$, then $b^{-1}a^{-1} = a^{-1}b^{-1}$ forces $G$ to be Abelian.

Finally, if $g$ is a group homomorphism, then

$$(ab)^2 = g(ab) = g(a)g(b) = a^2b^2$$

Now we multiply $(ab)^2 = a^2b^2$ by $a^{-1}$ by the left and $b^{-1}$ by the right, then we get $ba = ab$.

4. What is the order of $\sigma = (1\ 10\ 4)(2\ 13)(1\ 12\ 8)(5\ 7)(6\ 9)(5\ 11)$?

**Solution:** As the cycles are not disjoint we must first perform the multiplication of the cycles, we get

$$\sigma = (1\ 10\ 4)(2\ 13)(1\ 12\ 8)(5\ 7)(6\ 9)(5\ 11) = (1\ 12\ 8\ 10\ 4)(2\ 13)(5\ 11\ 7)(6\ 9)$$

Now that we have disjoint cycles we look at the order of each individual cycle, we get orders 5, 2, and 3. Thus the order of $\sigma$ is $2 \cdot 3 \cdot 5 = 30$ (as the orders are relatively prime).

5. Let

$$G = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \ ; \ a, b, c \in \mathbb{R},\ ac \neq 0 \right\}$$

Show that $G$ is a group under matrix multiplication, and that

$$H = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \ ; \ b \in \mathbb{R} \right\}$$

is a subgroup of $G$.

**Solution:** First let us check closure of $G$.

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{bmatrix} d & e \\ 0 & f \end{bmatrix} = \begin{bmatrix} ad & ae + bf \\ 0 & cf \end{bmatrix}$$

The product is in $G$ because its entries are in $\mathbb{R}$ and $(ad)(cf) \neq 0$, as $ac \neq 0$ and $df \neq 0$.

Note that if $a = c = d = f = 1$, then the diagonal entries of the product are also 1's. This shows closure of $H$.

Now note that

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}^{-1} = \begin{bmatrix} a^{-1} & -ba^{-1}c^{-1} \\ 0 & c^{-1} \end{bmatrix}$$

The inverse is in $G$ because its entries are in $\mathbb{R}$ and $a^{-1}c^{-1} \neq 0$, as $ac \neq 0$.

Note that if $a = c = 1$, then the diagonal entries of the inverse are also 1's. This shows the needed inverse property for $H$.

Associativity of $G$ follows directly from the associativity of $\mathbb{R}$.

Both $G$ and $H$ have identity equal to the identity $2 \times 2$ matrix. Hence, $G$ is a group and $H$ (clearly a subset of $G$) is a subgroup of $G$.

6. Let $G$ be a group and $H$ a subgroup of $G$. Let $N_H$ be the set of all $x \in G$ such that $xHx^{-1} = H$. Show that $N_H$ is a subgroup of $G$ containing $H$, and that $H$ is normal in $N_H$. (The group $N_H$ is called *the normalizer of* $H$.)

   **Solution:** First of all, if $x \in H$, then $xHx^{-1} = H$ because of the closure of $H$. So, $H \subset N_H$.

   Let $x, y \in N_H$, then

   $$(xy)H(xy)^{-1} = x\left(yHy^{-1}\right)x^{-1} = xHx^{-1} = H$$

   So, $N_H$ is closed.

   Clearly the identity of $G$ is in $H_N$.

   Now since $xHx^{-1} = H$ implies $x^{-1}Hx = H$ (because, for example, of the map $\varphi : G \to G$ defined by $\varphi(g) = xgx^{-1}$ being an isomorphism, and $\varphi^{-1}(g) = x^{-1}gx$).

   Finally, if $x \in N_H$ then $xHx^{-1} = H$, implying that $H$ is normal in $N_H$.

7. Define a relation $\sim$ on the set $\mathbb{N}$ of natural numbers by $a \sim b$ if and only if $a^2 + b^2$ is even. Prove that $\sim$ is an equivalence relation.

   **Solution:** We need to show that $\sim$ is symmetric, reflexive and transitive. The first two are clear as if $a^2 + b^2$ is even then $b^2 + a^2$ is even, and $a^2 + a^2 = 2a^2$ being even.

   Transitivity follows from the fact that the parity of $a^2 + c^2$ equals the parity of $a^2 + c^2 + 2b^2$, which is even, as it is the sum of the two even numbers $(a^2 + b^2)$ and $(b^2 + c^2)$.

8. Let $F$ be a field and $R$ any ring, and let $\varphi \colon F \to R$ be a nonzero ring homomorphism. Prove that $\ker(\varphi) = \{0\}$.

   **Solution:** Since the kernel of $\varphi$ is an ideal of the domain, then $ker(\varphi)$ is an ideal of the **field** $F$. Since the only ideals in a field are $\{0\}$ and the field itself, then $\varphi$ is either $1 - 1$ or the zero function. In this case, $\varphi$ is a non-zero homomorphism, thus $\ker(\varphi) = \{0\}$.

**Part B.** Solve **five** of the following eight problems :

1. Let $v_1, v_2, v_3$ be linearly independent vectors in a vector space $V$. Show that the vectors $v_1, 2v_1 + v_2, 3v_1 + 2v_2 + v_3$ are also linearly independent.

   **Solution:** A linear combination of the vectors $v_1, 2v_1+v_2, 3v_1+2v_2+v_3$ equal to zero yields

$$\begin{aligned} 0 &= \alpha v_1 + \beta(2v_1 + v_2) + \gamma(3v_1 + 2v_2 + v_3) \\ &= (\alpha + 2\beta + 3\gamma)v_1 + (\beta + 2\gamma)v_2 + \gamma v_3 \end{aligned}$$

   So, using that $v_1, v_2, v_3$ be linearly independent vectors we get the system

$$\begin{aligned} \alpha + 2\beta + 3\gamma &= 0 \\ \beta + 2\gamma &= 0 \\ \gamma &= 0 \end{aligned}$$

   which has solution $\alpha = \beta = \gamma = 0$. Hence, the vectors $v_1, 2v_1+v_2, 3v_1+2v_2+v_3$ are linearly independent.

2. Find explicitly a linear map $f : \mathbb{R}^2 \to \mathbb{R}^3$ such that $f(2,3) = (1,2,3)$ and $f(1,2) = (4,5,6)$. Is this linear function unique?

   **Solution:** First we write a generic element of $\mathbb{R}^2$ as a linear combination of the vectors $(2,3)$ and $(1,2)$,

$$\begin{aligned} (x,y) &= \alpha(2,3) + \beta(1,2) \\ &= (2\alpha + \beta, 3\alpha + 2\beta) \end{aligned}$$

   So, $2\alpha+\beta = x$ and $3\alpha+2\beta = y$. It follows that $\alpha = 2x - y$ and $\beta = -3x+2y$. Thus,

$$(x,y) = (2x-y)(2,3) + (-3x+2y)(1,2)$$

   So,

$$\begin{aligned} f(x,y) &= f\left[(2x-y)(2,3) + (-3x+2y)(1,2)\right] \\ &= f\left[(2x-y)(2,3)\right] + f\left[(-3x+2y)(1,2)\right] \\ &= (2x-y)f(2,3) + (-3x+2y)f(1,2) \\ &= (2x-y)(1,2,3) + (-3x+2y)(4,5,6) \\ &= (2x-y+4(-3x+2y), 2(2x-y)+5(-3x+2y), 3(2x-y)+6(-3x+2y)) \\ &= (-10x+7y, -11x+8y, -12x+9y) \end{aligned}$$

The construction of $f$ shows that the function is unique. Or, using that $\{(2,3),(1,2)\}$ is a basis of $\mathbb{R}^2$ then the function, being defined on a basis, must be unique.

3. Give examples of three of the following:

   (a) Three vectors in $\mathbb{R}^3$ so that any two are linear independent, but the set of all three is linearly dependent.

   (b) Matrices $M$ and $N$ such that $MN \neq NM$.

   (c) Linear maps
   $$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^4 \qquad\qquad g : \mathbb{R}^4 \longrightarrow \mathbb{R}^2$$
   such that $g \circ f$ is invertible.

   (d) A matrix $M$ such that $M^3 = 0$ but $M^2 \neq 0$.

**Solution:**

(a)
$$\{(1,0,0),(0,1,0),(1,1,0)\}$$

(b) Take, for example,
$$M = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad\qquad N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

(c) Consider
$$f(x,y) = (x,y,0,0) \qquad\qquad g(x,y,z,w) = (x,y)$$
Then $g \circ f$ is the identity.

(d) Consider
$$M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

4. Let $\mathcal{P}_2 = \{p(x) = a + bx + cx^2;\ a,b,c \in \mathbb{R}\}$. Show that the map $T : \mathcal{P}_2 \rightarrow \mathcal{P}_2$ given by
$$T(p) = \frac{d^2}{dx^2}\big((x^2+1)p(x)\big)$$
is linear. Then find the matrix of $T$ in the basis $\{1, x, x^2\}$.

**Solution:** Let us check linearity, we will use that the derivative is linear. Let $\alpha \in \mathbb{R}$ and $p(x), q(x) \in \mathcal{P}_2$.

$$\begin{aligned} T(\alpha p(x) + q(x)) &= \frac{d^2}{dx^2}\left[(x^2+1)(\alpha p(x) + q(x))\right] \\ &= \frac{d^2}{dx^2}\left[\alpha(x^2+1)p(x) + (x^2+1)q(x)\right] \\ &= \frac{d^2}{dx^2}\left[\alpha(x^2+1)p(x)\right] + \frac{d^2}{dx^2}\left[(x^2+1)q(x)\right] \\ &= \alpha\frac{d^2}{dx^2}\left[(x^2+1)p(x)\right] + \frac{d^2}{dx^2}\left[(x^2+1)q(x)\right] \\ &= \alpha T(p(x)) + T(q(x)) \end{aligned}$$

Now we look at the matrix in the standard basis

$$T(1) = \frac{d^2}{dx^2}\left(x^2+1\right) = 2$$

$$T(x) = \frac{d^2}{dx^2}\left(x^3+x\right) = 6x$$

$$T(x^2) = \frac{d^2}{dx^2}\left(x^4+x^2\right) = 12x^2+2$$

So, the matrix is given by

$$\begin{bmatrix} 2 & 0 & 2 \\ 0 & 6 & 0 \\ 0 & 0 & 12 \end{bmatrix}$$

5. Consider the following vector subspaces of $\mathbb{R}^3$

$$A = \{(x,y,z) \in \mathbb{R}^3 |\ 2x+y+3z = 0\},\ B = \text{span}\{(1,-2,0)\},\ C = \text{span}\{(1,1,-1)\}.$$

Show that $A$ is the *direct sum* of $B$ and $C$, that is $A = B \oplus C$.

**Solution:** First of all notice that $(1,-2,0)$ and $(0,-3,1)$ are linearly independent because one is not a multiple of the other. Also, since $A$ is the solution set f a homogeneous linear equation in $\mathbb{R}^3$ then it is a plane, which is two-dimensional. So, if both $(1,-2,0)$ and $(0,-3,1)$ belong to $A$ then the result follows.

Since both $(1,-2,0)$ and $(0,-3,1)$ clearly satisfy the equation $2x+y+3z=0$ then we are done.

6. Consider the subspace $U = \{(x, y, z) \in \mathbb{R}^3 | \, x + y - 2z = 0\} \subset \mathbb{R}^3$ and the system of vectors $S = \{v_1 = (1, 1, 1), v_2 = (5, -1, 2)\} \subset \mathbb{R}^3$.

   (a) Show that $U$ is the *span* of $S$.

   (b) Complete $S$ to a basis in $\mathbb{R}^3$.

   **Solution:**

   (a) Just like the previous problem we first notice that $v_1$ and $v_2$ are linearly independent. Secondly, both vectors satisfy the equation that defines $U$. Done.

   (b) Note that $(1, 0, 0)$ is not a solution of the equation that defines $U$, thus $(1, 0, 0)$ is not a linear combination of $v_1$ and $v_2$. So, $\{(1, 0, 0), v_1, v_2\}$ is a linearly independent set in $\mathbb{R}^3$, it follows it is a basis (three linearly independent vectors in a three-dimensional space).

7. Let $T$ be a linear operator on a vector space $V$, and let $x$ be an eigenvector of $T$ corresponding to the eigenvalue $\lambda$. For any positive integer $m$, prove that $x$ is an eigenvector of $T^m$ corresponding to the eigenvalue $\lambda^m$.

   **Solution:** We know $T(x) = \lambda x$. Note that

$$T^m(x) = (T^{m-1} \circ T)(x) = T^{m-1}(\lambda x) = \lambda T^{m-1}(x)$$

   The result follows by just generalizing what is above to $T^m(x) = \lambda^i \, T^{m-i}(x)$ for all $i$ (or by using induction).

8. If an $n \times n$ matrix $A$ is diagonalizable, show that $\det(A)$ is equal to the product of the eigenvalues of $A$.

   **Solution:** If $A$ is diagonalizable, then there is a matrix $P$ (change of basis) such that $PAP^{-1}$ is a diagonal matrix $D = diag(\lambda_1, \lambda_2, \cdots, \lambda_{n-1}, \lambda_n)$, where the $\lambda_i$'s are the eigenvalues of $A$.

   Since the determinant is a multiplicative function, then

$$det(D) = det\left(PAP^{-1}\right) = det(P)det(A)det\left(P^{-1}\right) = det(P)det\left(P^{-1}\right)det(A)$$

   But, since $det\left(P^{-1}\right) = det(P)^{-1}$, then $det(D) = det(A)$. The result follows from the fact that the determinant of a diagonal matrix is given by the product of its diagonal entries.