
Part A. Solve **five** of the following eight problems:

1. True or False: Prove or give a counterexample.
   Let a, b be non-trivial elements in a group G.
   (a) If $o(a) = m$ and $o(b) = m$, then $o(ab) = m$.
   (b) If $o(a) = m$ and $o(b) = n$ and $(m, n) = 1$, then $o(ab) = mn$.
   (c) If $o(a) = m$ then $o(a^{-1}) = m$.

2. Let N be a normal subgroup of G. Prove that G/N is abelian if and only if N contains all elements of the form $aba^{-1}b^{-1}$ for all $a, b \in G$.

3. Let R and S be rings, and let $\phi, \theta : R \to S$ be ring homomorphisms. Show that the set
   $$A = \{r \in R \mid \phi(r) = \theta(r)\}$$
   is a subring of R.

4. Let r be an element in an integral domain R such that $r^2 = r$.
   (a) Show that $(1-r)^2 = 1-r$ and that $r(1-r) = 0$.
   (b) Show that $rR \cap (1-r)R = \{0\}$.
   (c) Show that every element in R can be written as the sum of an element in rR plus an element in $(1-r)R$.

   You have just shown that $R = rR \oplus (1-r)R$.

5. Let $\sigma = (1234)(2345) \in S_5$. Find the index of $\langle\sigma\rangle$ in S_5.

6. Let R be a commutative ring and $a \in R$. The *annihilator* of a is defined by
   $$\mathrm{Ann}(a) = \{x \in R \mid xa = 0\}.$$
   Prove that $\mathrm{Ann}(a)$ is an ideal of R.

7. Consider the group $G = Mat_{2\times 2}(\mathbb{R})$ with the usual matrix addition. Let
   $$H = \{M \in G \mid \det(M) = 0\}.$$
   Is H is a subgroup of G? Prove your answer!

8. Show that $\mathbb{Z} \times \mathbb{Z}$ is not a cyclic group.

Part B is on the back!!!

Part B. Solve **five** of the following eight problems :

1. Determine conditions (if any) on b_1, b_2 and b_3 in order for the system

$$\begin{cases} x_1 + 3x_2 - 2x_3 &= b_1 \\ -x_1 - 5x_2 + 3x_3 &= b_2 \\ 2x_1 - 8x_2 + 3x_3 &= b_3 \end{cases}$$

to be consistent.

2. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let $v_1, v_2, \ldots, v_k$ be vectors in $\mathbb{R}^n$. If the set $\{T(v_1), T(v_2), \ldots, T(v_k)\}$ is linearly independent in $\mathbb{R}^m$, prove that the set $\{v_1, v_2, \ldots, v_k\}$ is linearly independent in $\mathbb{R}^n$.

3. Is the matrix $A = \begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix}$ diagonalizable? Justify your answer!

4. Given an $n \times n$ matrix A with $A^3 = 0$, show that $A - I_n$ is nonsingular.

5. Consider the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ determined by the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}.$$

   (a) Find a basis for the kernel of T.

   (b) Find a basis for the range (or image) of T.

6. Consider the linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ for which $f(1, -2) = (2, -1)$ and $f(3, -5) = (-3, 2)$. Find the matrix of f with respect to the standard basis. Then find the matrix of f with respect to the basis $B = \{(2, 3), (1, 2)\}$.

7. Consider the following vectors in $\mathbb{R}^4$:

$$v_1 = (1, 1, 0, 0)\ ,\ v_2 = (0, 1, 1, 0) \text{ and } v_3 = (0, 0, 1, 1)$$

Prove that $\{v_1, v_2, v_3\}$ is linearly independent and find an orthonormal basis for span$\{v_1, v_2, v_3\}$.

8. Let A be a 4×4 matrix with $det(A) = 2$ and rows v_1, v_2, v_3, and v_4. Find

$$\begin{vmatrix} v_1 \\ v_2 \\ 6v_3 + 5v_4 \\ 5v_3 + 9v_4 \end{vmatrix}$$