Algebra Qualifying Exam, Solutions Fall 2009

Part A.

1. True or False: Prove or give a counterexample.
Let a, b be non-trivial elements in a group G.

1. If $o(a) = m$ and $o(b) = m$, then $o(ab) = m$.
2. If $o(a) = m$ and $o(b) = n$ and $(m, n) = 1$, then $o(ab) = mn$.
3. If $o(a) = m$ then $o(a^{-1}) = m$.

Solution.

1. False: $o((12)(13)) = 3 \neq 2 = o((12)) = o((13))$.
2. False: $o((12)(134)) = 4 \neq 2 \cdot 3 = o((12))o((134))$.
3. True: If $a^m = e$, then $(a^{-1})^m = e$, which implies that $o(a^{-1})|m$, in particular $o(a^{-1})|o(a)$. Similarly $o(a)|o(a^{-1})|o(a)$.

2. Let N be a normal subgroup of G. Prove that G/N is abelian if and only if N contains all elements of the form $aba^{-1}b^{-1}$ for all $a, b \in G$.

Solution. Let $aN, bN \in G/N$ (which we know is a group because $N \trianglelefteq G$.
Assume that $aba^{-1}b^{-1} \in N$ for all $a, b \in G$, then

$$(aN)(bN)(aN)^{-1}(bN)^{-1} = (aba^{-1}b^{-1})N$$

Since, $aba^{-1}b^{-1} \in N$ then $(aN)(bN)(aN)^{-1}(bN)^{-1} = N$, which means that G/N is Abelian.
The converse uses the same idea.

3. Let R and S be rings, and let $\phi, \theta : R \to S$ be ring homomorphisms. Show that the set

$$A = \{r \in R \mid \phi(r) = \theta(r)\}$$

is a subring of R.

Solution. We just need to check a few properties.

- Closure for $+$ and inverses for $+$: Let $r, s \in A$, then $\phi(r) = \theta(r)$ and $\phi(s) = \theta(s)$. It follows that

$$\phi(r - s) = \phi(r) - \phi(s) = \theta(r) - \theta(s) = \theta(r - s)$$

- Clearly 0 is in A, as $\phi(0) = 0 = \theta(0)$.

- Closure for the multiplication:

$$\phi(rs) = \phi(r)\phi(s) = \theta(r)\theta(s) = \theta(rs)$$

- If your definition of ring includes the condition of containing a 1, then it is customary to also ask that homomorphisms send 1 to 1. Hence, 1 would be in A.

  If you consider rings without 1 then there is nothing to prove.

4. Let r be an element in an integral domain R such that $r^2 = r$.

1. Show that $(1 - r)^2 = 1 - r$ and that $r(1 - r) = 0$.
2. Show that $rR \cap (1 - r)R = \{0\}$.
3. Show that every element in R can be written as the sum of an element in rR plus an element in $(1 - r)R$.

You have just shown that $R = rR \oplus (1 - r)R$.

Solution.

1. Since $r^2 = r$, then

$$(1 - r)^2 = 1 - 2r + r^2 = 1 - 2r + r = 1 - r$$

   Similarly

$$r(1 - r) = r - r^2 = r - r = 0$$

2. Let $s \in rR \cap (1 - r)R$. Then $s = ra = (1 - r)b$ for some $a, b \in R$. Using the previous part we multiply both sides of

$$ra = (1 - r)b$$

   by r to obtain

$$r^2a = r(1 - r)b$$

   which is $ra = 0$. Since R is an integral domain, then $r = 0$ or $a = 0$. If $r = 0$ the result is trivial, if $a = 0$ then the intersection is just the zero element.

3. Let $s \in R$, then $s = rs + (1 - r)s$. Clearly $rs \in rR$ and $(1 - r)s \in (1 - r)R$.

Another way to approach this problem was by realizing that $r^2 = r$ has only two solutions in an integral domain: $r = 0$ and $r = 1$. This can be seen by transforming $r^2 = r$ into $r^2 - r = 0$, which is $r(r - 1) = 0$. After this the results asked to show are pretty trivial, as $\{r, (1 - r)\} = \{0, 1\}$.

5. Let $\sigma = (1234)(2345) \in S_5$. Find the index of $\langle \sigma \rangle$ in S_5.

Solution. Multiplying

$$\sigma = (1234)(2345) = (12453)$$

which is an element of order 5.
Hence, $< \sigma >$ has five elements. It follows that the index is

$$[S_5 :< \sigma >] = |S_5|/5 = 120/5 = 24$$

6. Let R be a commutative ring and $a \in R$. The *annihilator* of a is defined by

$$\text{Ann}(a) = \{x \in R \mid xa = 0\}.$$

Prove that $\text{Ann}(a)$ is an ideal of R.

Solution. We just check what is necessary.

- Let $r, s \in Ann(a)$, then $ra = sa = 0$. It follows that

$$(r - s)a = ra - sa = 0 - 0 = 0$$

- Clearly 0 is in $Ann(a)$, as $0 \cdot a = 0$.

- Finally, if $r \in Ann(a)$ (meaning $ra = 0$) and $s \in R$ then,

$$(sr)a = s(ra) = s(0) = 0$$

  Note that we just need to check $sr \in Ann(a)$ as R **is commutative**.

7. Consider the group $G = Mat_{2\times 2}(\mathbb{R})$ with the usual matrix addition. Let

$$H = \{M \in G \mid \det(M) = 0\}.$$

Is H is a subgroup of G? Prove your answer!

Solution. No, it is not. For example, the following two matrices live in H

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad\qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

but their sum is the identity, which is not in H.

8. Show that $\mathbb{Z} \times \mathbb{Z}$ is not a cyclic group.

Solution. Assume that $\mathbb{Z} \times \mathbb{Z} =< (a, b) >$, where $a, b \in \mathbb{Z}$.
It is clear that if $a = 0$ then the element $(1, 0)$ is not in $< (a, b) >$, a contradiction.
Similarly, $b \neq 0$. Hence $ab \neq 0$.
Since the elements in $< (a, b) >$ look like (ax, bx), where $x \in \mathbb{Z}$, then if one of the coordinates of the element (ax, bx) is zero then so must be the other one. It follows that elements such as $(1, 0)$ are not in $< (a, b) >$.

Part B.

1. Determine conditions (if any) on b_1, b_2 and b_3 in order for the system

$$\begin{cases} x_1 + 3x_2 - 2x_3 &= b_1 \\ -x_1 - 5x_2 + 3x_3 &= b_2 \\ 2x_1 - 8x_2 + 3x_3 &= b_3 \end{cases}$$

to be consistent.

Solution. Adding equation one to equation two and subtracting twice equation one from equation three we get

$$\begin{cases} x_1 + 3x_2 - 2x_3 &= b_1 \\ -2x_2 + x_3 &= b_2 + b_1 \\ -14x_2 + 7x_3 &= b_3 - 2b_1 \end{cases}$$

Now we subtract seven times equation two from equation three to get

$$\begin{cases} x_1 + 3x_2 - 2x_3 &= b_1 \\ -2x_2 + x_3 &= b_2 + b_1 \\ 0 &= b_3 - 2b_1 - 7(b_2 + b_1) \end{cases}$$

So, in order for the system to be consistent we need $0 = b_3 - 2b_1 - 7(b_2 + b_1)$ or $b_3 = 9b_1 + 7b_2$.

2. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let $v_1,\ v_2,\ \ldots,\ v_k$ be vectors in $\mathbb{R}^n$. If the set $\{T(v_1),\ T(v_2),\ \ldots,\ T(v_k)\}$ is linearly independent in $\mathbb{R}^m$, prove that the set $\{v_1,\ v_2,\ \ldots,\ v_k\}$ is linearly independent in $\mathbb{R}^n$.

Solution. Assume that $\{T(v_1),\ T(v_2),\ \ldots,\ T(v_k)\}$ is linearly independent and that there are (not all zero) $\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_k \in \mathbb{R}$ such that

$$0 = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k$$

Applying T to the previous equation we get

$$\begin{aligned} T(0) &= T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k) \\ 0 &= T(\alpha_1 v_1) + T(\alpha_2 v_2) + \cdots + T(\alpha_k v_k) \\ 0 &= \alpha_1 T(v_1) + \alpha_2 T(v_2) + \cdots + \alpha_k T(v_k) \end{aligned}$$

which contradicts that $\{T(v_1),\ T(v_2),\ \ldots,\ T(v_k)\}$ is a linearly independent set.

3. Is the matrix $A = \begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix}$ diagonalizable? Justify your answer!

Solution. The characteristic polynomial of A is

$$\begin{aligned} \chi_A(\lambda) &= \det \begin{bmatrix} 3-\lambda & 1 \\ -4 & -1-\lambda \end{bmatrix} \\ &= (3-\lambda)(-1-\lambda) - 1(-4) \\ &= \lambda^2 - 2\lambda + 1 \end{aligned}$$

which has a double root $\lambda = 1$. Since the eigenvalues are not all distinct, then we need to look at the eigenspace of $\lambda = 1$ to determine whether or not A is diagonalizable. So, we set look at the equation $Ax = x$, or $(A - Id)x = 0$. Since

$$A - Id = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$$

which has rank one, it follows that A is not diagonalizable.
Another (simpler) way to see that A is not diagonalizable is to realize that if it were then it would be conjugated to the identity, since the conjugacy class of I has only one element then A is not diagonalizable.

4. Given an $n \times n$ matrix A with $A^3 = 0$, show that $A - I_n$ is nonsingular.

Solution. We take $A^3 = 0$ and we subtract I both sides, we get

$$(A - I)(A^2 + A + I) = A^3 - I = -I$$

Now taking determinants both sides

$$det(A - I)det(A^2 + A + I) = det(-I)$$

Since $det(-I) \neq 0$ then $det(A - I) \neq 0$.

5. Consider the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ determined by the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}.$$

1. Find a basis for the kernel of T.
2. Find a basis for the range (or image) of T.

Solution. Let us perform some row operations in this matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -24 \end{bmatrix} \quad \text{we did } R_2 \mapsto R_2 - 5R_1 \text{ and } R_3 \mapsto R_3 - 9R_1$$

$$\to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{we did } R_3 \mapsto R_3 - 2R_2$$

$$\to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{we did } R_2 \mapsto -1/4R_2$$

$$\to \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{we did } R_1 \mapsto R_1 - 2R_2$$

Let us call the latter matrix A. It is clear that A has the same kernel as the original matrix and that it has the same rank (not range necessarily!).

1. In order to find the kernel we set $Ax = 0$. We get

   $$x - z - 2w = 0 \qquad y + 2z + 3w = 0$$

   or

   $$x = z + 2w \qquad y = -2z - 3w$$

   which has solution space

   $$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} z + 2w \\ -2z - 3w \\ z \\ w \end{bmatrix} = z \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

   It follows that

   $$Ker(matrix) = span \left(\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right)$$

2. Since the rank is two then we know we need two linearly independent column vectors of the original matrix to span the range. So, for instance,

   $$Range(matrix) = span \left(\begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 10 \end{bmatrix} \right)$$

6. Consider the linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ for which $f(1,-2) = (2,-1)$ and $f(3,-5) = (-3,2)$. Find the matrix of f with respect to the standard basis. Then find the matrix of f with respect to the basis $B = \{(2,3),(1,2)\}$.

Solution. From the info given we get

$$f(1,0) - 2f(0,1) = (2,-1) \qquad\qquad 3f(1,0) - 5f(0,1) = (-3,2).$$

which yields (three times the first equation minus the second)

$$-f(0,1) = 3(2,-1) - (-3,2).$$

or $f(0,1) = (-9,5)$. Similarly, $f(1,0) = (-16,9)$.
It follows that the matrix of f in the standard basis $\mathcal{C}$ is

$$[f]_{\mathcal{C}} = \begin{bmatrix} -16 & -9 \\ 9 & 5 \end{bmatrix}$$

In order to get the matrix with respect to the basis $B = \{(2,3),(1,2)\}$ we need the change of basis matrices. The change of basis from B to $\mathcal{C}$ is

$$[E]_{B\mapsto\mathcal{C}} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

Hence, the matrix of f in base B is

$$\begin{aligned}
[f]_B &= \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -16 & -9 \\ 9 & 5 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} -16 & -9 \\ 9 & 5 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \\
&= \begin{bmatrix} -151 & -87 \\ 243 & 140 \end{bmatrix}
\end{aligned}$$

7. Consider the following vectors in $\mathbb{R}^4$:

$$v_1 = (1,1,0,0)\ ,\ v_2 = (0,1,1,0) \text{ and } v_3 = (0,0,1,1)$$

Prove that $\{v_1, v_2, v_3\}$ is linearly independent and find an orthonormal basis for $\text{span}\{v_1, v_2, v_3\}$.

Solution. Setting

$$\alpha_1(1,1,0,0)+\alpha_2(0,1,1,0)+\alpha_3(0,0,1,1)=(0,0,0,0)$$

we get

$$\alpha_1=0 \qquad \alpha_1+\alpha_2=0 \qquad \alpha_2+\alpha_3=0 \qquad \alpha_3=0$$

So, the set is linearly independent.
Now we use Gram-Schmidt to get an orthogonal basis of $V=\text{span}\{v_1,v_2,v_3\}$, then we divide by norms to get all the norms to be one.

$$w_1=v_1=(1,1,0,0)$$

$$\begin{aligned} w_2=v_2-\frac{<v_1,v_2>}{<v_1,v_1>}v_1 &= (0,1,1,0)-\frac{<(1,1,0,0),(0,1,1,0)>}{<(1,1,0,0),(1,1,0,0)>}(1,1,0,0) \\ &= (0,1,1,0)-\frac{1}{2}(1,1,0,0) \\ &= \left(-\frac{1}{2},\frac{1}{2},1,0\right) \end{aligned}$$

$$\begin{aligned} w_3=v_3-\frac{<v_1,v_3>}{<v_1,v_1>}v_1-\frac{<w_2,v_3>}{<w_2,w_2>}w_2 &= (0,0,1,1)-\frac{<(1,1,0,0),(0,0,1,1)>}{<(1,1,0,0),(1,1,0,0)>}(1,1,0,0)- \\ &\quad \frac{<\left(-\frac{1}{2},\frac{1}{2},1,0\right),(0,0,1,1)>}{<\left(-\frac{1}{2},\frac{1}{2},1,0\right),\left(-\frac{1}{2},\frac{1}{2},1,0\right)>}\left(-\frac{1}{2},\frac{1}{2},1,0\right) \\ &= (0,0,1,1)-(0,0,0,0)-\frac{2}{3}\left(-\frac{1}{2},\frac{1}{2},1,0\right) \\ &= \left(\frac{1}{3},-\frac{1}{3},\frac{1}{3},1\right) \end{aligned}$$

It follows that the orthonormal basis we found is

$$\left\{ \frac{1}{\sqrt{2}}(1,1,0,0), \sqrt{\frac{2}{3}}\left(-\frac{1}{2},\frac{1}{2},1,0\right), \frac{\sqrt{3}}{2}\left(\frac{1}{3},-\frac{1}{3},\frac{1}{3},1\right) \right\}$$

8. Let A be a 4×4 matrix with $det(A)=2$ and rows v_1,v_2,v_3, and v_4. Find

$$\begin{vmatrix} v_1 \\ v_2 \\ 6v_3+5v_4 \\ 5v_3+9v_4 \end{vmatrix}$$

Solution. Using that the determinant is multilinear we get (working on the third row)

$$\begin{vmatrix} v_1 \\ v_2 \\ 6v_3 + 5v_4 \\ 5v_3 + 9v_4 \end{vmatrix} = 6 \begin{vmatrix} v_1 \\ v_2 \\ v_3 \\ 5v_3 + 9v_4 \end{vmatrix} + 5 \begin{vmatrix} v_1 \\ v_2 \\ v_4 \\ 5v_3 + 9v_4 \end{vmatrix}$$

Now working on both fourth rows we get

$$6 \begin{vmatrix} v_1 \\ v_2 \\ v_3 \\ 5v_3 + 9v_4 \end{vmatrix} + 5 \begin{vmatrix} v_1 \\ v_2 \\ v_4 \\ 5v_3 + 9v_4 \end{vmatrix} = 30 \begin{vmatrix} v_1 \\ v_2 \\ v_3 \\ v_3 \end{vmatrix} + 54 \begin{vmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{vmatrix} + 25 \begin{vmatrix} v_1 \\ v_2 \\ v_4 \\ v_3 \end{vmatrix} + 45 \begin{vmatrix} v_1 \\ v_2 \\ v_4 \\ v_4 \end{vmatrix}$$

Since determinants with repeated rows must be zero and $det(A) = 2$ we get

$$\begin{vmatrix} v_1 \\ v_2 \\ 6v_3 + 5v_4 \\ 5v_3 + 9v_4 \end{vmatrix} = 30 \cdot 0 + 54 \cdot 2 - 25 \cdot 2 + 45 \cdot 0 = 108 - 50 = 58$$

Note that we used the property of determinants of getting a negative sign every time we switch the position of two rows.