
Part A. Solve **five** of the following eight problems:

1. Let G be a group.
 - (a) If $a, b \in G$ and $ab = e$, prove that $ba = e$.
 - (b) If $|G| = 4$, prove that G is Abelian.
 - (c) If $a, b \in G$, prove that $o(bab^{-1}) = o(a)$.

 2. Let $\sigma \in S_n$ be a permutation. For $x, y \in \{1, 2, \dots, n\}$ let $x \sim y$ if and only if $\sigma^k(x) = y$ for some $k \in \mathbb{Z}$. Show that \sim is an equivalence relation on the set $\{1, 2, \dots, n\}$.

Hint. Note that σ is fixed; it is k that changes.

 3. Let $R = \left\{ \begin{bmatrix} a & 2b \\ b & a \end{bmatrix} ; a, b \in \mathbb{Z} \right\}$ and $S = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$. Define $\varphi: R \rightarrow S$ by $\varphi\left(\begin{bmatrix} a & 2b \\ b & a \end{bmatrix}\right) = a + b\sqrt{2}$. Prove that φ is a ring isomorphism. *You may assume that R is a commutative ring with identity using the usual matrix addition and multiplication, that S is a commutative ring with identity using the usual addition and multiplication of real numbers.*

 4. True or False (prove or give a counterexample)
 - (a) The elements in \mathbb{Z}_n are either zero, invertible or zero divisors.
 - (b) Part (a) is true for all rings.

 5. Let G be a group of (finite) order n , and $\phi: G \rightarrow \mathbb{C}^*$ a homomorphism of groups. Prove that $\phi(g)$ is an n^{th} root of 1 (and thus, lays on the unit circle), for all $g \in G$.

 6. How many elements of order 7 does the group $\mathbb{Z}_{21} \oplus \mathbb{Z}_{35}$ have? Justify your answer.

 7. (a) Let R be a commutative ring with $a \in R$. The *annihilator* of a is defined by
$$\text{Ann}(a) = \{x \in R \mid xa = 0\}.$$

Prove that $\text{Ann}(a)$ is an ideal of R .

 - (b) Show that the direct sum of two nonzero rings is never an integral domain.

 8. Recall that the *center* of a group G is the subset $Z(G) = \{a \in G \mid ax = xa \text{ for all } x \in G\}$.
 - (a) Prove that $Z(G)$ is a subgroup of G ;
 - (b) Prove that $Z(G)$ is a normal subgroup of G .
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Part B is on the back!!!

Part B. Solve **five** of the following eight problems :

1. Let W be the subspace of \mathbb{R}^4 with basis $\left\{ \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$. Use the Gram-Schmidt process to find an orthonormal basis for W .

2. Show that $A \in M_{nn}$ is invertible if and only if 0 is not an eigenvalue of A .

3. Consider the linear transform $T : P^2 \rightarrow P^2$ defined by

$$T(a + bx + cx^2) = (a + b + c) + 2(b + c)x + 3cx^2.$$

Find the matrix of T with respect to the basis $\{1, 1 + x, 3 + 4x + 2x^2\}$.

4. Let U be the subspace of \mathbb{R}^4 spanned by $\{(1, 0, 1, 0), (-1, 2, 0, 1)\}$ and let V be the subspace of \mathbb{R}^4 spanned by $\{(0, 2, 1, 1), (0, 0, 1, 1)\}$.

- (a) Find a basis of $U \cap V$.
(b) Extend (a) to a basis for U .

5. Give the rank and nullity of the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 & -2 & 5 \\ 0 & 1 & -1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

6. Suppose that W is a vector space with dimension 5, and U and V are subspaces of W , each of which has dimension 3. Prove that $U \cap V$ contains a nonzero vector.

7. Diagonalize

$$\begin{pmatrix} 3 & -2 & 2 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}.$$

8. Let α, β, γ be real numbers. Evaluate the determinant

$$\begin{vmatrix} \sin^2 \alpha & \sin^2 \beta & \sin^2 \gamma \\ \cos^2 \alpha & \cos^2 \beta & \cos^2 \gamma \\ 1 & 1 & 1 \end{vmatrix}$$
