
Part A. Solve **five** of the following eight problems:

1. Let G and H be groups.

(a) Prove that

$$S(G, H) = \{f : G \rightarrow H; f \text{ is a function}\}$$

is a group with the operation $*$ defined by

$$(f * g)(x) = f(x) \star g(x)$$

for all $x \in G$, where \star is the operation in H .

(b) Prove that $S(G, H)$ is abelian if and only if H is abelian.

2. (a) Can there be a group homomorphism from $\mathbb{Z}_6 \oplus \mathbb{Z}_6$ onto \mathbb{Z}_{12} ?

(b) Can there be a group homomorphism from \mathbb{Z}_{81} onto $\mathbb{Z}_3 \oplus \mathbb{Z}_3$?

3. (a) Let a, b and c be elements of a commutative ring with unity, and suppose that a is a unit. Prove that b divides c in this ring if and only if ab divides c in this ring.

(b) Let a belong to a ring R . Let $S = \{x \in R \mid ax = 0\}$. Show that S is a subring of R .

4. Let G, H be groups and $\phi : G \rightarrow H$ be a homomorphism.

(a) Prove that $\phi(g^i) = \phi(g)^i$, for all $i \in \mathbb{Z}$.

(b) Show that if ϕ is an isomorphism and G is cyclic, then H is cyclic.

5. Prove the following claims for permutations in S_n .

(a) $(a_1 a_2 \cdots a_k) = (a_1 a_k) \cdots (a_1 a_3)(a_1 a_2) = (a_1 a_2)(a_2 a_3) \cdots (a_{k-1} a_k)$, where a_1, a_2, \dots, a_k are distinct.

Conclude that every permutation is the product of transpositions.

(b) Any permutation can be written as a product of the transpositions $(12), (13), \dots, (1n)$.

6. Let $S = \mathbb{R} \setminus \{-1\}$. Define $*$ on S by $a * b = a + b + ab$. Show that S is a group.

7. Show, in full detail, that $I = \{p(x) \in \mathbb{R}[x]; p(0) = p(1) = 0\}$ is a principal ideal of $\mathbb{R}[x]$.

8. Let d be a positive integer. Prove that $\mathbb{Q}[\sqrt{d}] = \{a + b\sqrt{d} \mid a, b \in \mathbb{Q}\}$ is a field.

Part B is on the back!!!

Part B. Solve **five** of the following eight problems :

1. Let \mathbb{P}_2 be the vector space of real polynomials of degree 2 or less. Consider the basis $B = \{1, x, x^2\}$ and $B' = \{1, x + 1, x^2 + x + 1\}$ for \mathbb{P}_2 .

(a) Find the transition matrix from B to B' .

(b) Let $p(x) = 3 - x + 2x^2 \in \mathbb{P}_2$. Find the coordinates of $p(x)$ relative to B' .

2. Find an orthonormal basis for the subspace of \mathbb{R}^4 spanned by the vectors $\mathbf{u}_1 = (0, 0, -1, 0)$, $\mathbf{u}_2 = (1, 3, 0, 0)$, and $\mathbf{u}_3 = (1, 0, 1, 1)$.

3. Find a square root of the matrix

$$A = \begin{bmatrix} 1 & 3 & -3 \\ 0 & 4 & 5 \\ 0 & 0 & 9 \end{bmatrix}.$$

4. Let V be an n -dimensional vector space and W a subspace of V . Let $B' = \{b_1, b_2, \dots, b_m\}$ be a basis for W . Prove that there exist vectors $b_{m+1}, b_{m+1}, \dots, b_n$ in V such that $B = \{b_1, b_2, \dots, b_m, b_{m+1}, \dots, b_n\}$ is a basis for V .

5. Gaussian elimination changes $A\mathbf{x} = \mathbf{b}$ to a reduced $R\mathbf{x} = \mathbf{d}$. The complete solution is

$$\mathbf{x} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}.$$

(a) What is the reduced row echelon form matrix R ?

(b) What is \mathbf{d} ?

6. Consider the linear transformation with matrix

$$A = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Find a basis for the kernel and a basis for the image of the transformation.

7. Let A be a nonzero $n \times n$ skew-symmetric real matrix.

(a) Show that if n is odd, then A is not invertible.

(b) What happens if n is even? Justify your answer.

8. (a) Show that if \mathbf{v} is an eigenvector of the matrix product AB and $B\mathbf{v} \neq \mathbf{0}$, then $B\mathbf{v}$ is an eigenvector of BA .

(b) Suppose that \mathbf{v} is an eigenvector of A corresponding to an eigenvalue λ . Show that \mathbf{v} is an eigenvector of $5I - 3A + A^2$, where I is the identity matrix of the same size as A . What is the corresponding eigenvalue?
