Mathematics Department Qualifying Exam : Algebra Spring 2003

Part A. Solve **five** of the following eight problems :

1. Prove that if $N \trianglelefteq G$ and H is any subgroup of G, then $N \cap H \trianglelefteq H$.

2. Prove that if H and K are finite subgroups of G whose orders are relatively prime, then $H \cap K = \{e\}$

3. Show that the relation on $\mathbb{Z}$ defined by $a \sim b$ iff $a^2 \equiv b^2 \mod 6$ is an equivalence relation.

4. Suppose that ϕ is a homomorphism from $\mathbb{Z}_{30}$ to $\mathbb{Z}_{30}$ and $ker(\phi) = \{0, 10, 20\}$. If $\phi(23) = 9$, determine all elements that map to 9. That is, find all $k \in \mathbb{Z}_{30}$ such that $\phi(k) = 9$.

5. (a) Show that $a = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ has order 3 in $GL(2, \mathbb{R})$ and $b = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ has order 4.
   (b) Show that ab has infinite order.

6. Prove that σ^2 is an even permutation for every permutation σ.

7. Define a new addition and multiplication on $\mathbb{Z}$ by

   $$a \oplus b = a + b - 1 \qquad \text{and} \qquad a \otimes b = ab - (a + b) + 2$$

   Prove that with these operations $\mathbb{Z}$ is an integral domain.

8. Show that a finite commutative ring with no zero-divisors has a multiplicative identity.

Part B. Solve **five** of the following eight problems :

1. Suppose that A is an $n \times n$ matrix such that $A^3 = 0$ but $A^2 \neq 0$. Show that $\{I, A, A^2\}$ is independent in the space of all $n \times n$ matrices with real entries.

2. Let A be diagonalizable 2×2 matrix. If $\lambda^4 = 5\lambda$ for each eigenvalue λ of A, show that $A^4 = 5A$.

3. If $T : V \to V$ is linear, show that $T^2 = I_V$ iff T is an isomorphism and $T^{-1} = T$.

4. (a) Find A if $(A^{-1} - 3I)^T = 2\begin{pmatrix} -1 & 2 \\ 5 & 4 \end{pmatrix}$

   (b) If $\det(A) = 2$ and $\det(B) = -3$, compute $\det(A^3 B^{-1} A^T B^2)$.

5. Invertible 2×2 matrices with determinant one have the form
   $\begin{pmatrix} a & b \\ c & \frac{1+bc}{a} \end{pmatrix}$, where $a \neq 0$ and $bc \neq 1$, or
   $\begin{pmatrix} 0 & b \\ -\frac{1}{b} & d \end{pmatrix}$, where $b \neq 0$ and $d \neq 0$, or
   $\begin{pmatrix} a & b \\ -\frac{1}{b} & 0 \end{pmatrix}$, where $b \neq 0$
   Determine the form(s) of all 2×2 matrices that are their own inverses.

6. Let $V = \{v \in \mathbb{R} | \ v > 0\}$. Show that V is a vector space over $\mathbb{R}$ if the vector addition is ordinary multiplication and scalar multiplication is defined by $a \cdot v = v^a$.

7. If $\mathbb{R}^n = span\{v_1, v_2, \cdots, v_n\}$ and if x and y in $\mathbb{R}^n$ satisfy $x \cdot v_i = y \cdot v_i$ for all i, show that $x = y$.

8. Let P_n denote the space of all polynomials with real coefficients that have degree at most n (union the zero polynomial). Find a linear transformation $T : P_2 \to P_4$ such that

$$T(1) = x^4, \qquad T(1+x) = 1 + x^3 \qquad T(1+x^2) = 1 - x^2$$