Algebra Qualifying Exam, Solutions Spring 2003

Part A.

1. Prove that if $N \trianglelefteq G$ and H is any subgroup of G, then $N \cap H \trianglelefteq H$.

   Solution. We know the intersection of two subgroups of G is a subgroup of G. It follows that $N \cap H < H$.

   Let $x \in N \cap H$, and $h \in H$. Since N is normal in G and $H \subset G$, then $hxh^{-1} \in N$. Moreover, by closure in H $hxh^{-1} \in H$. So, $hxh^{-1} \in N \cap H$.

2. Prove that if H and K are finite subgroups of G whose orders are relatively prime, then $H \cap K = \{e\}$

   Solution. This is problem 7 in part A in the exam of Fall 2006.

3. Show that the relation on $\mathbb{Z}$ defined by $a \sim b$ iff $a^2 \equiv b^2 \mod 6$ is an equivalence relation.

   Solution. Since $a^2 \equiv a^2 \mod 6$, and 6 dividing $a^2 - b^2$ implies that 6 divides $b^2 - a^2$ then the only thing left to check is transitivity. So, assume $a^2 \equiv b^2 \mod 6$ and $b^2 \equiv c^2 \mod 6$, that is $a^2 - b^2 = 6x$ and $b^2 - c^2 = 6y$ for some integers x, y. Then,

   $$a^2 - c^2 = (a^2 - b^2) + (b^2 - c^2) = 6x + 6y = 6(x + y)$$

   Since $x + y \in \mathbb{Z}$ then we are done.

4. Suppose that ϕ is a homomorphism from $\mathbb{Z}_{30}$ to $\mathbb{Z}_{30}$ and $ker(\phi) = \{0, 10, 20\}$. If $\phi(23) = 9$, determine all elements that map to 9. That is, find all $k \in \mathbb{Z}_{30}$ such that $\phi(k) = 9$.

Solution. We know that the pre-image of any element in the range of a homomorphism is a coset of the kernel of the homomorphism. So, in this case,

$$\phi^{-1}(9) = 23 + ker(\phi) = \{23, 23 + 10, 23 + 20\} = \{23, 33, 43\}$$

5. (a) Show that $a = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ has order 3 in $GL(2, \mathbb{R})$ and $b = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ has order 4.

   (b) Show that ab has infinite order.

   Solution.

   (a) Easy computations show

   $$a^2 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \qquad a^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

   and

   $$b^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad b^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

   (b) This follows from

   $$ab = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \qquad \text{and thus} \qquad (ab)^k = \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix}$$

   for all $k \in \mathbb{Z}_+$

6. Prove that σ^2 is an even permutation for every permutation σ.

   Solution. This follows from the fact that in the integers

   $$even \ + \ even = odd \ + \ odd = even$$

   In this case the number of 2-cycles in the representation of σ^2 is the sum of the number of 2-cycles in σ, so it is the sum of two even or two odd numbers.

7. Define a new addition and multiplication on $\mathbb{Z}$ by

   $$a \oplus b = a + b - 1 \qquad \text{and} \qquad a \otimes b = ab - (a + b) + 2$$

   Prove that with these operations $\mathbb{Z}$ is an integral domain.

Solution. We first check that $(\mathbb{Z}, \oplus, \otimes)$ is a ring.

Closure is clear, it is also clear that the additive identity is the number 1, and the additive inverse of an element a is $2 - a$. Note that this ring (to be) has no multiplicative identity.

Just checking at the formulas for $\oplus$ and $\otimes$ we see that both operations are 'commutative'. So, we have ourselves a commutative ring.

Now assume that for $a, b \in \mathbb{Z}$ we have

$$0 = a \otimes b$$

which means

$$1 = ab - (a + b) + 2$$

and this implies

$$(a - 1)(b - 1) = 0$$

Hence, either $a = 1$ or $b = 1$. Since 1 is the additive identity in our ting, then we are done.

8. Show that a finite commutative ring with no zero-divisors has a multiplicative identity.

Solution. This is problem 1 in part A of the exam of Spring 2006.

Part B.

1. Suppose that A is an $n \times n$ matrix such that $A^3 = 0$ but $A^2 \neq 0$. Show that $\{I, A, A^2\}$ is independent in the space of all $n \times n$ matrices with real entries.

   Solution. If $\{I, A, A^2\}$ is linearly dependent, then there is a non-trivial linear combination of then that equals zero, that is

   $$aA^2 + bA + cI = 0$$

   for some $a, b, c \in \mathbb{R}$.

   But this implies that the minimal polynomial of A, $\mu_A(x)$, divides $p(x) = ax^2 + bx + c$. However, $A^3 = 0$ implies that $\mu_A(x) = x, x^2$ or x^3 and, since the degree of $p(x)$ is two, then $\mu_A(x) = x, x^2$, which contradicts the assumption $A^2 \neq 0$.

2. Let A be diagonalizable 2×2 matrix. If $\lambda^4 = 5\lambda$ for each eigenvalue λ of A, show that $A^4 = 5A$.

   Solution. Since A is diagonalizable, then there is a matrix P such that $PAP^{-1} = D$, where D is a diagonal matrix with diagonal entries the eigenvalues of A.

   Since $\lambda^4 = 5\lambda$ for each eigenvalue λ of A, it is easy to see that $D^4 = 5D$. But this implies that so does $A = P^{-1}DP$, in fact

   $$A^4 = (P^{-1}DP)^4 = P^{-1}D^4P = P^{-1}(5D)P = 5(P^{-1}DP) = 5A$$

3. If $T : V \to V$ is linear, show that $T^2 = I_V$ iff T is an isomorphism and $T^{-1} = T$.

   Solution. $T^2 = I$ means that the composition of T with itself is the identity. Hence T is its own inverse, and thus it is an isomorphism.

   The other direction is immediate.

4. (a) Find A if $(A^{-1} - 3I)^T = 2\begin{pmatrix} -1 & 2 \\ 5 & 4 \end{pmatrix}$

   (b) If $\det(A) = 2$ and $\det(B) = -3$, compute $\det(A^3B^{-1}A^TB^2)$.

Solution.

(a) Transposing we get

$$(A^{-1} - 3I) = \begin{pmatrix} -2 & 10 \\ 4 & 8 \end{pmatrix}$$

then, adding $3I$ both sides we get

$$A^{-1} = \begin{pmatrix} 1 & 10 \\ 4 & 11 \end{pmatrix}$$

It follows that

$$A^{-1} = \frac{1}{11-40} \begin{pmatrix} 11 & -10 \\ -4 & 1 \end{pmatrix} = \frac{1}{29} \begin{pmatrix} -11 & 10 \\ 4 & -1 \end{pmatrix}$$

(b)

$$\begin{aligned} \det(A^3 B^{-1} A^T B^2) &= \det(A^3)\det(B^{-1})\det(A^T)\det(B^2) \\ &= \det(A)^3 \det(B)^{-1} \det(A)\det(B)^2 \\ &= \det(A)^4 \det(B) \\ &= 2^4(-3) = -48 \end{aligned}$$

5. Invertible 2×2 matrices with determinant one have the form
$\begin{pmatrix} a & b \\ c & \frac{1+bc}{a} \end{pmatrix}$, where $a \neq 0$ and $bc \neq 1$, or
$\begin{pmatrix} 0 & b \\ -\frac{1}{b} & d \end{pmatrix}$, where $b \neq 0$ and $d \neq 0$, or
$\begin{pmatrix} a & b \\ -\frac{1}{b} & 0 \end{pmatrix}$, where $b \neq 0$
Determine the form(s) of all 2×2 matrices that are their own inverses.

Solution. The short way to solve this would be to realize that $A^2 = I$ means that the minimal polynomial of A divides $x^2 - 1$. It follows that

$\mu_A(x) = x+1$ or $\mu_A(x) = x-1$ or $\mu_A(x) = (x-1)(x+1)$

In any case, the minimal polynomial of A has distinct roots, thus A is diagonalizable. It follows that A diagonalizes to

$$B = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$$

So, the answer would be any conjugation of the four matrices B in the previous equation. Of course $\pm Id$ do not yield anything interesting, the others, though, give a big family of matrices.

6. Let $V = \{v \in \mathbb{R} | \ v > 0\}$. Show that V is a vector space over $\mathbb{R}$ if the vector addition is ordinary multiplication and scalar multiplication is defined by $a \cdot v = v^a$.

   Solution. Fix $v, w \in \mathbb{R}_{>0}$ and $a, b \in \mathbb{R}$.

   Since both vw and v^a are positive reals, then closure works out. Also,

   $$a(v + w) = (vw)^a = v^a w^a = av + aw$$

   and

   $$(a + b)v = v^{a+b} = v^a v^b = av + aw$$

   Finally, 0 in this vector space is the number 1, as $v + 1 = v \cdot 1 = v$. It follows that the additive inverse of v in V is v^{-1}, which is also a positive real number.

7. If $\mathbb{R}^n = span\{v_1, v_2, \cdots, v_n\}$ and if x and y in $\mathbb{R}^n$ satisfy $x \cdot v_i = y \cdot v_i$ for all i, show that $x = y$.

   Solution. First note that since the spanning set has the same number as the dimension of the space, then the set is a basis. Now let

   $x = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n$ and $x = y_1 v_1 + y_2 v_2 + \cdots + y_n v_n$

   Then

   $$x \cdot v_i = x_1(v_1 \cdot v_i) + x_2(v_2 \cdot v_i) + \cdots + x_n(v_n \cdot v_i)$$

   and

   $$y \cdot v_i = y_1(v_1 \cdot v_i) + y_2(v_2 \cdot v_i) + \cdots + y_n(v_n \cdot v_i)$$

So,

$$y_1(v_1 \cdot v_i) + y_2(v_2 \cdot v_i) + \cdots + y_m(v_m \cdot v_i) = x_1(v_1 \cdot v_i) + x_2(v_2 \cdot v_i) + \cdots + x_n(v_n \cdot v_i)$$

or

$$(x_1 - y_1)(v_1 \cdot v_i) + (x_2 - y_2)(v_2 \cdot v_i) + \cdots + (x_n - y_n)(v_n \cdot v_i) = 0$$

for all i.

But this implies

$$[(x_1 - y_1)v_1 + (x_2 - y_2)v_2 + \cdots + (x_n - y_n)v_n] \cdot v_i = 0$$

for all i.

So, the vector $w = (x_1 - y_1)v_1 + (x_2 - y_2)v_2 + \cdots + (x_n - y_n)v_n$ is orthogonal to all vectors in the spanning set of $\mathbb{R}^n$. It follows that $w = 0$. Since w is a linear combination of linearly independent vectors, then $x_i = y_i$ for all i, and thus $x = y$

8. Let P_n denote the space of all polynomials with real coefficients that have degree at most n (union the zero polynomial). Find a linear transformation $T : P_2 \to P_4$ such that

$$T(1) = x^4, \qquad T(1 + x) = 1 + x^3 \qquad T(1 + x^2) = 1 - x^2$$

Solution. Since T is linear, then

$$T(x) = T(1 + x - 1) = T(1 + x) - T(1) = 1 + x^3 - x^4$$

and

$$T(x^2) = T(1 + x^2 - 1) = T(1 + x^2) - T(1) = 1 - x^2 - x^4$$

It follows that

$$\begin{aligned} T(a + bx + cx^2) &= T(a) + T(bx) + T(cx^2) \\ &= aT(1) + bT(x) + cT(x^2) \\ &= a(x^4) + b(1 + x^3 - x^4) + c(1 - x^2 - x^4) \\ &= (b + c) - cx^2 + bx^3 + (a - b - c)x^4 \end{aligned}$$