Mathematics Department Qualifying Exam Spring 2004

Subject: Algebra

Part A. Do **five** of the following 8 problems.

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1. Let a, b, m, and n be integers, and suppose $am + bn = 1$. Prove that a and b are relatively prime.

2. Let $T = \mathbb{R}^3 - \{(0,0,0)\}$. Define a relation $\sim$ on T by $(x_1, y_1, z_1) \sim (x_2, y_2, z_2)$ if and only if there exists a nonzero real number λ such that $x_1 = \lambda x_2$, $y_1 = \lambda y_2$, and $z_1 = \lambda z_2$. Prove that $\sim$ is an equivalence relation.

3. Let G and H be groups, and let $\varphi\colon G \to H$ be an *onto* group homomorphism. Suppose G is abelian. Prove that H is abelian.

4. Let G be a group, and let N be the subset $\{g \in G \mid gx = xg \text{ for all } x \in G\}$ (N is called the *center* of G). Prove that N is a normal subgroup of G.

5. Let S_n denote the group of permutations on the set $\{1, 2, \ldots, n\}$, and let A_n denote the subset consisting of even permutations.

   (a) Prove that A_n is a normal subgroup of S_n. *You may assume A_n is a subgroup of S_n.*

   (b) Prove that S_n/A_n is isomorphic to the group $\mathbb{Z}_2 = \{0, 1\}$.

6. Let R be a ring with identity element 1_R, and let I be an ideal of R. Prove that if 1_R is in I, then $I = R$.

7. Let $\varphi\colon \mathbb{C} \to \mathbb{C}$ be defined by $\varphi(a + bi) = a - bi$ for all $a + bi \in \mathbb{C}$. Prove that φ is a ring isomorphism.

8. Prove that the only ideals of a field F are $\{0_F\}$ and F, where 0_F denotes the additive identity element of F.

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Part B. Do **five** of the following 8 problems.

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1. Let $A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 3 \end{bmatrix}$, and let R be the reduced row echelon form for A.

   (a) Find R, determine the (row) rank of A, and find a basis for the row space of A.

   (b) Find a matrix P such that $PA = R$.

2. Find an orthonormal basis for the subspace of the Euclidean space $\mathbb{R}^3$ spanned by the vectors $v_1 = (1, 0, 1)$ and $v_2 = (0, 3, 4)$.

3. Prove: If S is a finite linearly independent subset of the vector space V and $w \in V$ is not in the subspace spanned by S, then the set $S \cup \{w\}$ is linearly independent.

4. Let V and W be vector spaces over the field F and let T be a linear transformation from V into W. Suppose V is finite dimensional. Prove: rank (T) + nullity (T) = dim (V).

5. For each natural number n, determine the value of the determinant of the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 1 & 0 & 3 & 4 & \cdots & n \\ 1 & 2 & 0 & 4 & \cdots & n \\ \vdots & & & \ddots & \cdots & \vdots \\ 1 & 2 & 3 & 4 & \cdots & 0 \end{bmatrix}$$

6. Let A be the symmetric matrix $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. Find an orthogonal matrix T such that $T^{-1}AT$ is a diagonal matrix.

7. Let $A = (a_{ij})$ be an $n \times n$ matrix over the reals. Show that A can be expressed in a unique way as $A = S + K$ where S is symmetric and K is skew-symmetric. *(Hint: Consider the matrices $\frac{1}{2}(A + A^t)$ and $\frac{1}{2}(A - A^t)$).*

8. Let A and B be $n \times n$ matrices and suppose A and B are similar. Show:

   (a) $\det(A) = \det(B)$.

   (b) If A is nonsingular, so is B, and A^{-1} is similar to B^{-1}.