Algebra Qualifying Exam, Solutions Spring 2004

Part A.

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1. Let a, b, m, and n be integers, and suppose $am+bn=1$. Prove that a and b are relatively prime.

   Solution. Suppose $d=(a,b)$. Let $a=ds$, $b=dt$. Then $1=am+bn=dsm+dtn=d(sm+tn)$. Therefore $d=1$.

2. Let $T=\mathbb{R}^3-\{(0,0,0)\}$. Define a relation $\sim$ on T by $(x_1,y_1,z_1)\sim(x_2,y_2,z_2)$ if and only if there exists a nonzero real number λ such that $x_1=\lambda x_2$, $y_1=\lambda y_2$, and $z_1=\lambda z_2$. Prove that $\sim$ is an equivalence relation.

   Solution. An equivalence relation must be reflexive, symmetric and transitive.

   Reflexive. Let $\lambda=1$. Then $(x,y,z)=1\cdot(x,y,z)$, so $(x,y,z)\sim(x,y,z)$.

   Symmetric. Suppose $(x_1,y_1,z_1)=\lambda(x_2,y_2,z_2)$. Then $(x_2,y_2,z_2)=\frac{1}{\lambda}(x_1,y_1,z_1)$.

   Transitive. Suppose $(x_1,y_1,z_1)=\lambda(x_2,y_2,z_2)$ and $(x_2,y_2,z_2)=\mu(x_3,y_3,z_3)$. Then $(x_1,y_1,z_1)=\lambda(\mu(x_3,y_3,z_3))=\lambda\mu(x_3,y_3,z_3)$.

3. Let G and H be groups, and let $\varphi\colon G\to H$ be an *onto* group homomorphism. Suppose G is abelian. Prove that H is abelian.

   Solution. Let h, $k\in H$. Since φ is onto there exist a, $b\in G$ such that $\varphi(a)=h$ and $\varphi(b)=k$. Thus $hk=\varphi(a)\varphi(b)=\varphi(ab)=\varphi(ba)=\varphi(b)\varphi(a)=kh$.

4. Let G be a group, and let N be the subset $\{g\in G\mid gx=xg \text{ for all } x\in G\}$ (N is called the *center* of G). Prove that N is a normal subgroup of G.

   Solution 1. First note that $1\in N$ since $1\cdot x=x\cdot 1$ for all $x\in G$. Thus N is nonempty. Let m and n be elements of N, and let $x\in G$. Since $xn=nx$, we have $x=nxn^{-1}$ and thus $n^{-1}x=xn^{-1}$. Therefore $xmn^{-1}=mxn^{-1}=mn^{-1}x$, and $mn^{-1}\in N$. Thus $N\leq G$.

   To show that N is normal, let m and x be as above. Then $xm=mx$ and thus $xmx^{-1}=m\in N$.

   Solution 2. First note that $1\in N$ since $1\cdot x=x\cdot 1$ for all $x\in G$. Now let m and n be elements of N, and let $x\in G$. Then $xmn=mxn=mnx$; therefore $mn\in N$. Since $xn=nx$, we have $x=nxn^{-1}$ and thus $n^{-1}x=xn^{-1}$. Therefore $n^{-1}\in N$. Thus $N\leq G$.

   To show that N is normal, let m and x be as above. Then $xm=mx$ and thus $xmx^{-1}=m\in N$.

5. Let S_n denote the group of permutations on the set $\{1, 2, \ldots, n\}$, and let A_n denote the subset consisting of even permutations.

   (a) Prove that A_n is a normal subgroup of S_n. *You may assume A_n is a subgroup of S_n.*

   (b) Prove that S_n/A_n is isomorphic to the group $\mathbb{Z}_2 = \{0, 1\}$.

   Solution.

   (a) Let $\tau \in A_n$ and let $\sigma \in S_n$. If σ is even, then so is σ^{-1}. Similarly, if σ is odd, then so is σ^{-1}. Therefore $\sigma\tau\sigma^{-1}$ can be written as an even number of transpositions.

   (b) **Solution 1.** Let $\varphi\colon S_n \to \mathbb{Z}_2$ be defined by $\varphi(\sigma) = \begin{cases} 0 & \text{if } \sigma \text{ is even} \\ 1 & \text{if } \sigma \text{ is odd} \end{cases}$.

   Clearly φ is an onto homomorphism since even $\circ$ even = even, odd $\circ$ odd = even, etc. Moreover, $\ker(\varphi) = A_n$. Therefore by the First Isomorphism Theorem, $S_n/A_n \cong \mathbb{Z}_2$.

   Solution 2. Let $\varphi\colon S_n/A_n \to \mathbb{Z}_2$ be defined by $\varphi(\sigma A_n) = \begin{cases} 0 & \text{if } \sigma \text{ is even} \\ 1 & \text{if } \sigma \text{ is odd} \end{cases}$.

   Clearly φ is an onto homomorphism since even $\circ$ even = even, odd $\circ$ odd = even, etc. Moreover, $|S_n/A_n| = 2 = |\mathbb{Z}_2|$. Therefore φ is an isomorphism.

6. Let R be a ring with identity element 1_R, and let I be an ideal of R. Prove that if 1_R is in I, then $I = R$.

   Solution. Let $r \in R$. Then since $1_R \in I$ we have $r \cdot 1_R = r \in I$ since I is an ideal of R.

7. Let $\varphi\colon \mathbb{C} \to \mathbb{C}$ be defined by $\varphi(a + bi) = a - bi$ for all $a + bi \in \mathbb{C}$. Prove that φ is a ring isomorphism.

   Solution. Clearly φ is onto. φ is also one-to-one since if $\varphi(a + bi) = a - bi = 0$, then $a = 0$ and $b = 0$, and thus $a + bi = 0$. φ is a ring homomorphism since

   - $\varphi((a + bi) + (c + di) = \varphi(a + c + (b + d)i) = a + c - (b + d)i = a - bi + c - di = \varphi(a + bi) + \varphi(c + di)$
   - $\varphi((a + bi)(c + di) = \varphi(ac - bd + (ad + bc)i) = ac - bd - (ad + bc)i = (a - bi)(c - di) = \varphi(a + bi)\varphi(c + di)$.

   Therefore φ is a ring isomorphism.

8. Prove that the only ideals of a field F are $\{0_F\}$ and F, where 0_F denotes the additive identity element of F.

   Solution. Suppose I is an ideal of F. If $I = \{0_F\}$ then we are done, so suppose $a \in I$, $a \neq 0$. Then there is an element $a^{-1} \in F$ since F is a field. Therefore $a^{-1}a = 1 \in I$ since I is an ideal. But by #6, this implies $I = F$.

Part B.

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1. Let $A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 3 \end{bmatrix}$, and let R be the reduced row echelon form for A.

   (a) Find R, determine the (row) rank of A, and find a basis for the row space of A.

   (b) Find a matrix P such that $PA = R$.

   Solution. We need to perform row reduction on A to get to R. The matrices of the elementary operations we will use will yield P.

$$
\begin{aligned}
A &= \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 3 \end{bmatrix} && \text{now we do } R_1 \mapsto R_1 - \frac{1}{3}R_3 \\
&\to \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 3 \end{bmatrix} && \text{now we do } R_2 \mapsto R_2 - \frac{4}{3}R_3 \\
&\to \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} = R
\end{aligned}
$$

   It follows that

$$
P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & 1 \end{bmatrix}
$$

2. Find an orthonormal basis for the subspace of the Euclidean space $\mathbb{R}^3$ spanned by the vectors $v_1 = (1, 0, 1)$ and $v_2 = (0, 3, 4)$.

   Solution. The first step on the Gram-Schmidt process says that we fix v_1 and define

$$
\begin{aligned}
u_2 &= (0,3,4) - \frac{(1,0,1)\cdot(0,3,4)}{|(1,0,1)|^2}(1,0,1) \\
&= (0,3,4) - 2(1,0,1) \\
&= (-2,3,2)
\end{aligned}
$$

   So, right now we have an orthogonal basis. We obtain an orthonormal basis by dividing v_1 and u_2 by their norm. The final answer is

$$
\left\{ \frac{1}{\sqrt{2}}(1,0,1),\ \frac{1}{\sqrt{17}}(-2,3,2) \right\}
$$

3. Prove: If S is a finite linearly independent subset of the vector space V and $w \in V$ is not in the subspace spanned by S, then the set $S \cup \{w\}$ is linearly independent.

Solution. Consider a linear combination of the elements of S and w that is equal to zero. If the scalar with w is zero, then we have a linear combination of the elements of S that is equal to zero, thus all the scalars are equal to zero. If the scalar with w is different from zero, then we can 'solve' for w and leave w as a linear combination of the elements of S, which would imply that $w \in Span(S)$. A contradiction.

4. Let V and W be vector spaces over the field F and let T be a linear transformation from V into W. Suppose V is finite dimensional. Prove: rank (T) + nullity (T) = dim (V).

Solution. We know that $V/ker(T) \cong Im(T)$. So, $\dim(V/ker(T) = \dim(Im(T)) = rank(T)$.

Since

$$\dim(V/ker(T)) = \dim(V) - \dim(ker(T)) = \dim(V) - nullity(T)$$

then

$$\dim(V) - nullity(T) = rank(T)$$

5. For each natural number n, determine the value of the determinant of the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 1 & 0 & 3 & 4 & \cdots & n \\ 1 & 2 & 0 & 4 & \cdots & n \\ \vdots & & & \ddots & \cdots & \vdots \\ 1 & 2 & 3 & 4 & \cdots & 0 \end{bmatrix}$$

Solution.

$$\begin{aligned} \det(A) &= \begin{vmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 1 & 0 & 3 & 4 & \cdots & n \\ 1 & 2 & 0 & 4 & \cdots & n \\ \vdots & & & \ddots & \cdots & \vdots \\ 1 & 2 & 3 & 4 & \cdots & 0 \end{vmatrix} && \text{now we subtract row 1 to all rows} \\ &= \begin{vmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 0 & -2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -3 & 0 & \cdots & 0 \\ \vdots & & & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -n \end{vmatrix} \\ &= (-1)^{n-1} n! && \text{because the matrix above is upper triangular} \end{aligned}$$

6. Let A be the symmetric matrix $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. Find an orthogonal matrix T such that $T^{-1}AT$ is a diagonal matrix.

Solution. Since A is symmetric, then its eigenvectors associated to distinct eigenvalues must be orthogonal. So, the only thing we have to do is to get the standard diagonalization of A.

The characteristic polynomial of A is

$$\chi_A(\lambda) = (1-\lambda)^2 - 1 = \lambda^2 - 2\lambda = \lambda(\lambda - 2)$$

For $\lambda = 0$ we get the system

$$x - y = 0 \qquad\qquad -x + y = 0$$

which has solution space spanned by $(1,1)$.

For $\lambda = 2$ we get the system

$$x - y = 2x \qquad\qquad -x + y = 2y$$

which has solution space spanned by $(1,-1)$.

It follows that $T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

7. Let $A = (a_{ij})$ be an $n \times n$ matrix over the reals. Show that A can be expressed in a unique way as $A = S + K$ where S is symmetric and K is skew-symmetric. *(Hint: Consider the matrices $\frac{1}{2}(A + A^t)$ and $\frac{1}{2}(A - A^t)$).*

Solution. Note that the matrix $\frac{1}{2}(A + A^t)$ is symmetric and that the matrix $\frac{1}{2}(A - A^t)$ is skew-symmetric. Also, note that

$$\frac{1}{2}(A + A^t) + \frac{1}{2}(A - A^t) = A$$

Now we just have to check the uniqueness of the representation. So, we assume there are matrices M (symmetric) and N (skew-symmetric) such that $A = M + N$. So,

$$M + N = \frac{1}{2}(A + A^t) + \frac{1}{2}(A - A^t)$$

which implies

$$M - \frac{1}{2}(A + A^t) = \frac{1}{2}(A - A^t) - N$$

In the previous equation the right hand side is skew-symmetric and the left hand side is symmetric. It follows that

$$M - \frac{1}{2}(A + A^t) = \frac{1}{2}(A - A^t) - N = 0$$

Hence, the representation is unique.

8. Let A and B be $n \times n$ matrices and suppose A and B are similar. Show:

   (a) $\det(A) = \det(B)$.

   (b) If A is nonsingular, so is B, and A^{-1} is similar to B^{-1}.

   Solution. Assume $B = PAP^{-1}$

   (a) Since the determinant is multiplicative, then

   $$\det(B) = \det(PAP^{-1}) = \det(P)\det(A)\det(P^{-1}) = \det(P)\det(A)\det(P)^{-1} = \det(A)$$

   (b) If A is nonsingular, then $\det(A) \neq 0$, which implies $\det(B) \neq 0$, and thus B is non-singular. In this case

   $$B^{-1} = (PAP^{-1})^{-1} = (P^{-1})^{-1}A^{-1}P^{-1} = PA^{-1}P^{-1}$$

   So, A^{-1} is similar to B^{-1}.