Mathematics Department Qualifying Exam Spring 2005
Subject: Algebra

Part A. Do **five** of the following 8 problems.

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1. Let a and p be integers. If p is prime and a is not divisible by p, prove that the additive order of a modulo p is equal to p.

2. Let G and H be groups, and let $\varphi\colon G \to H$ be a group homomorphism with kernel $\ker(\varphi)$. Prove that $\ker(\varphi)$ is a normal subgroup of G.

3. Let N be a normal subgroup of a group G. Prove that the factor group G/N is abelian if and only if $aba^{-1}b^{-1} \in N$ for all elements $a,\ b \in G$.

4. Let G be any group with no proper nontrivial subgroups, and assume the order of G is greater than 1. Prove that G is finite cyclic of order p for some prime p.

5. Let G be a group and let $D = \{(a, a, a) \mid a \in G\}$.
   (a) Prove that D is a subgroup of the direct product $G \times G \times G$.
   (b) Prove that D is normal in $G \times G \times G$ if and only if G is abelian.
   Hint. If D is normal, then $(a, a, b)(b, b, b)(a, a, b)^{-1} \in D$ for all $a,\ b \in G$.

6. Let $\mathbb{Q}[x]$ be the set of all polynomials in x with rational coefficients. Define a relation $\sim$ on $\mathbb{Q}[x]$ by $f(x) \sim g(x)$ if and only if $f(x) - g(x)$ is divisible by $x^2 + 1$. Prove that $\sim$ is an equivalence relation.

7. Let R be the ring $\{m+n\sqrt{2} \mid m, n \in \mathbb{Z}\}$, and let I be the subset $\{m+n\sqrt{2} \in R \mid m \text{ is even}\}$. Prove that I is an ideal of R.

8. Assume that the set $S = \{a + b\sqrt{3} \mid a,\ b \text{ are rational numbers}\}$ is a commutative ring. Prove that S is a field.

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Part B. Do **five** of the following 8 problems.

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1. For which values of the parameters a, b, and c is the matrix $A = \begin{bmatrix} a & 1 & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$ invertible? Find the inverse when it exists.

2. Consider the linear transformation with matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 3 & 2 \end{bmatrix}$. Find a basis for the kernel and a basis for the image of the transformation.

3. Let $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$. Find all eigenvalues of A and all their corresponding eigenvectors.

4. Find an orthonormal basis for the subspace of $\mathbb{R}^3$ spanned by the vectors $v_1 = \langle 1, 0, -1 \rangle$ and $v_2 = \langle 0, 3, 4 \rangle$

5. (a) Show that two non-zero vectors are linearly dependent if and only if one is a scalar multiple of the other.

   (b) Let v_1, v_2, and v_3 be linearly independent vectors in $\mathbb{R}^n$. Are the vectors v_1, v_2, and $v_1 + v_2 + v_3$ necessarily linearly independent?

6. Let $\mathbb{C}$ denote the field of complex numbers and $\mathbb{R}$ the field of real numbers. With the usual operations, $\mathbb{C}$ is a vector space over $\mathbb{R}$. Prove that the map $\varphi \colon \mathbb{C} \to \mathbb{R}^2$ given by $\varphi(x + iy) = (x, y)$ is an isomorphism of vector spaces.

7. Prove or disprove: The matrix

$$A = \begin{bmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{bmatrix}$$

   over $\mathbb{R}$ has determinant equal to $(y - x)(z - x)(z - y)$.

8. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Prove that $A^n = 2^{n-1}A$ for all positive integers n.