Mathematics Department Qualifying Exam Spring 2006
Subject: Algebra

Part A. Do **five** of the following 8 problems.

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1. Show that a nonzero finite commutative ring R with no zero-divisors has a multiplicative identity.

2. Let $G = \{x \in \mathbb{R} |\ x > 0 \text{ and } x \neq 1\}$. Define the operation $*$ on G by $a * b = a^{\ln b}$, for all $a, b \in G$. Prove that G is an Abelian group under the operation $*$.

3. Let $\mathbb{R}[x]$ denote the ring of all polynomials with real coefficients. Also, let $a \in \mathbb{R}$, and let $f(x) \in \mathbb{R}[x]$, with derivative $f'(x)$. Show that the remainder when $f(x)$ is divided by $(x-a)^2$ is $f'(a)(x-a) + f(a)$.

4. Let $\phi : G_1 \to G_2$ and $\theta : G_2 \to G_3$ be group homomorphisms. Prove that $\theta\phi : G_1 \to G_3$ is a homomorphism. prove that $ker(\phi) \subset ker(\theta\phi)$.

5. Let N be a subgroup of the center of G. Show that if G/N is a cyclic group, then G must be Abelian.

6. Show that a relation on $\mathbb{R}^+$ defined by $x \sim y$ iff $x^y = y^x$ is an equivalence relation.

7. Let F be a field and let $\phi : F \to R$ be a ring homomorphism. Show that ϕ is either zero or one-to-one.

8. Let S_n denote the symmetric group of degree n and let A_n denote the alternating group of degree n. For any elements $\sigma, \tau \in S_n$ show that $\sigma\tau\sigma^{-1}\tau^{-1} \in A_n$.

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Part B. Do **five** of the following 8 problems.

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1. Consider the linear transformation with matrix $A = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Find a basis for the kernel and a basis for the image of the transformation.

2. Let A be an $m \times n$ matrix. Consider the set
$$W = \left\{ \mathbf{v} \in \mathbb{R}^n \;\middle|\; A\mathbf{v} = \mathbf{0} \right\}$$
Prove that W a subspace of $\mathbb{R}^n$.

3. Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Prove that the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ represents a rotation of the vector x by an angle of θ.

4. Find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by the vectors $u_1 = \langle 0, -1, 0, 0 \rangle$, $u_2 = \langle 3, 0, 1, 0 \rangle$, and $u_3 = \langle 1, 1, 0, 1 \rangle$.

5. Let A be an invertible $n \times n$ matrix, and let c be a nonzero real number. Prove or disprove: $(cA)^{-1} = \frac{1}{c} A^{-1}$.

6. Let $A_{m\times n}$ and $B_{n\times p}$ be matrices. Prove that $(AB)^T = B^T A^T$, where A^T denotes the transpose of the matrix A.

7. Let $A = \begin{bmatrix} 0 & 0 & -2 \\ 0 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$. Find the eigenvalues and corresponding eigenvector(s) of A.

8. Let A be an invertible $n \times n$ matrix. Prove that $\det(A^{-1}) = \dfrac{1}{\det(A)}$.