Algebra Qualifying Exam, Solutions Spring 2006

Part A.

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1. Show that a nonzero finite commutative ring R with no zero-divisors has a multiplicative identity.

   Solution. Fix an element $r \in R^*$. Consider the map $\phi : R \to R$ defined by $\phi(x) = xr$. Note that

   $$\phi(x) = \phi(y) \qquad \text{implies} \qquad r(x-y) = 0$$

   Since R has no zero divisors, then ϕ is one-to-one. But, as R is finite then ϕ is bijective. It follows there is an element $x_r \in R$ such that $rx_r = r$.

   using that ϕ is onto we represent an element $y \in R$ as $y = rx$ for some $x \in R$, then

   $$x_r y = x_r(rx) = (x_r r)x = rx = y$$

   So, x_r works as the identity for ALL elements in R.

2. Let $G = \{x \in \mathbb{R} |\ x > 0 \text{ and } x \neq 1\}$. Define the operation $*$ on G by $a * b = a^{\ln b}$, for all $a, b \in G$. Prove that G is an Abelian group under the operation $*$.

   Solution. Let $a, b \in G$, then $a^{\ln b}$ is also a positive real number (a is positive). Moreover, since $\ln b \neq 0$ (as $b \neq 1$) then $a^{\ln b} \neq 1$. So, closure for the product works out.

   What is the identity? It is e!! (meaning $e \sim 2.71$)... confusing as the standard notation for the identity in a group is also e. To check this just notice that

   $$e^{\ln b} = b \qquad \text{and} \qquad a^{\ln e} = a^1 = a$$

   What is the inverse of $a \in G$? We look for an element b such that $a * b = e$, or in other words

   $$a^{\ln b} = e$$

   Applying natural log both sides we get

   $$\ln\left(a^{\ln b}\right) = \ln e$$

   which is

   $$\ln a \ln b = 1$$

   So, b is given by the unique positive number such that $\ln b$ is the multiplicative inverse of $\ln a$.

   Finally, we check that the group is Abelian

   $$a * b = a^{\ln b} = (e^{\ln a})^{\ln b} = (e^{\ln b})^{\ln a} = b^{\ln a}$$

3. Let $\mathbb{R}[x]$ denote the ring of all polynomials with real coefficients. Also, let $a \in \mathbb{R}$, and let $f(x) \in \mathbb{R}[x]$, with derivative $f'(x)$. Show that the remainder when $f(x)$ is divided by $(x-a)^2$ is $f'(a)(x-a) + f(a)$.

Solution. Since the degree of $f'(a)(x-a)+f(a)$ is one, then we just need to show that

$$f(x) = q(x)(x-a)^2 + f'(a)(x-a) + f(a)$$

for some $q(x) \in \mathbb{R}[x]$.

The remainder theorem says that

$$f(x) = p(x)(x-a) + f(a)$$

for some $p(x) \in \mathbb{R}[x]$.

we now derive the previous equation both sides to get

$$f'(x) = p'(x)(x-a) + p(x)$$

So, $f'(a) = p(a)$

We use the remainder theorem again with $p(x)$ divided by $(x-a)$ to get

$$p(x) = q(x)(x-a) + p(a)$$

for some $q(x) \in \mathbb{R}[x]$.

It follows that

$$\begin{aligned} f(x) &= p(x)(x-a) + f(a) \\ &= [q(x)(x-a) + p(a)](x-a) + f(a) \\ &= q(x)(x-a)^2 + p(a)(x-a) + f(a) \\ &= q(x)(x-a)^2 + f'(a)(x-a) + f(a) \end{aligned}$$

4. Let $\phi : G_1 \longrightarrow G_2$ and $\theta : G_2 \longrightarrow G_3$ be group homomorphisms. Prove that $\theta\phi : G_1 \longrightarrow G_3$ is a homomorphism. prove that $ker(\phi) \subset ker(\theta\phi)$.

   Solution. Let $g, h \in G_1$

   $$\theta\phi(gh) = \theta(\phi(gh)) = \theta(\phi(g)\phi(h)) = \theta(\phi(g))\theta(\phi(h)) = \theta\phi(g)\theta\phi(h)$$

   Hence, $\theta\phi$ is a group homomorphism

   Now let $g \in ker(\phi)$, then

   $$\theta\phi(g) = \theta(\phi(g)) = \theta(e_{G_2}) = e_{G_3}$$

   So, $g \in ker(\theta\phi)$.

5. Let N be a subgroup of the center of G. Show that if G/N is a cyclic group, then G must be Abelian.

Solution. Let $G/N =< gN >$. Take two elements in G, $x = g^i n$ and $y = g^j m$, where $i, j \in \mathbb{Z}$ and $n, m \in N$, then (don't forget that both n and m live in the center of G)

$$\begin{aligned} xy &= (g^i n)(g^j m) \\ &= g^i (n g^j) m \\ &= g^i (g^j n) m \\ &= (g^i g^j)(nm) \\ &= (g^j g^i)(mn) \\ &= g^j (g^i m) n \\ &= g^j (m g^i) n \\ &= (g^j m)(g^i n) = yx \end{aligned}$$

6. Show that a relation on $\mathbb{R}^+$ defined by $x \sim y$ iff $x^y = y^x$ is an equivalence relation.

   Solution. Since $x^y = y^x$ implies $y^x = x^y$ and $x^x = x^x$ then $\sim$ is both reflexive and symmetric.

   Now assume that $x^y = y^x$ and $y^z = z^y$ (note that none of these elements is zero), then

   $$x^z = (x^y)^{z/y} = (y^x)^{z/y} = (y^z)^{x/y} = (z^y)^{x/y} = z^x$$

   Hence, transitivity also holds.

7. Let F be a field and let $\phi : F \longrightarrow R$ be a ring homomorphism. Show that ϕ is either zero or one-to-one.

   Solution. Since F is a field, then it has no proper ideals. However, the kernel of ϕ is an ideal of F, thus $ker(\phi) = \{0\}$ (in which case ϕ is one-to-one) or $ker(\phi) = F$, in the latter case ϕ is the zero function.

8. Let S_n denote the symmetric group of degree n and let A_n denote the alternating group of degree n. For any elements $\sigma, \tau \in S_n$ show that $\sigma\tau\sigma^{-1}\tau^{-1} \in A_n$.

   Solution. Note that $\sigma\tau$ is even if and only if $\sigma^{-1}\tau^{-1}$ is even. The result follows by noting that the product of two permutations having the same parity is always even.

Part B.

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1. Consider the linear transformation with matrix $A = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Find a basis for the kernel and a basis for the image of the transformation.

   Solution. Set $A\mathbf{x} = \mathbf{0}$. Then the solution space is $\left\{ \begin{bmatrix} 2z \\ -3z \\ z \\ 0 \end{bmatrix} \middle| z \in \mathbb{R} \right\}$. Therefore it is of dimension 1, and a basis is $\left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix} \right\}$.

   Since A is 3×4, the image is a subspace of $\mathbb{R}^3$. Moreover, the dimension of the image is 3 since the dimension of the kernel is 1. A 3-dimensional subspace of $\mathbb{R}^3$ must be $\mathbb{R}^3$ itself, so we may use the standard basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

2. Let A be an $m \times n$ matrix. Consider the set

   $$W = \{ \mathbf{v} \in \mathbb{R}^n \mid A\mathbf{v} = \mathbf{0} \}$$

   Prove that W a subspace of $\mathbb{R}^n$.

   Solution. Let $\mathbf{u}$, $\mathbf{v} \in W$ and let $c \in \mathbb{R}$. Then

   (i) $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}$. Therefore $\mathbf{u} + \mathbf{v} \in W$.

   (ii) $A(c\mathbf{u}) = cA\mathbf{u} = c \cdot \mathbf{0} = \mathbf{0}$. Therefore $c\mathbf{u} \in W$.

   Since W is closed under addition and scalar multiplication, W is a subspace of $\mathbb{R}^n$.

3. Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Prove that the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ represents a rotation of the vector x by an angle of θ.

   Solution. Let $\mathbf{a} = \begin{bmatrix} x \\ y \end{bmatrix}$ and let $\mathbf{b} = A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta x - \sin\theta y \\ \sin\theta x + \cos\theta y \end{bmatrix}$. We must show that $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$.

We have $|\mathbf{a}| = \sqrt{x^2 + y^2}$ and

$$\begin{aligned} |\mathbf{b}| &= \sqrt{(x\cos\theta - y\sin\theta)^2 + (x\sin\theta + y\cos\theta)^2} \\ &= \sqrt{x^2\cos^2\theta - 2xy\cos\theta\sin\theta + y^2\sin^2\theta + x^2\sin^2\theta + 2xy\sin\theta\cos\theta + y^2\cos^2\theta} \\ &= \sqrt{x^2 + y^2}. \end{aligned}$$

Therefore

$$\begin{aligned} \mathbf{a}\cdot\mathbf{b} &= x^2\cos\theta - xy\sin\theta + xy\sin\theta + y^2\cos\theta \\ &= (x^2 + y^2)\cos\theta \\ &= |\mathbf{a}||\mathbf{b}|\cos\theta. \end{aligned}$$

4. Find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by the vectors $u_1 = \langle 0, -1, 0, 0\rangle$, $u_2 = \langle 3, 0, 1, 0\rangle$, and $u_3 = \langle 1, 1, 0, 1\rangle$.

   Solution. Using the Gram-Schmidt process, we have that v_1, v_2, v_3 is an orthogonal basis, where $v_1 = u_1 = \langle 0, -1, 0, 0\rangle$, $v_2 = u_2 = \langle 3, 0, 1, 0\rangle$ (since $u_1 \cdot u_2 = 0$), and $v_3 = \langle 1, 1, 0, 1\rangle - \frac{-1}{1}\langle 0, -1, 0, 0\rangle - \frac{3}{10}\langle 3, 0, 1, 0\rangle = \langle \frac{1}{10}, 0, -\frac{3}{10}, 1\rangle$.
   We have $|v_1| = 1$, $|v_2| = \sqrt{10}$, and $|v_3| = \sqrt{110}$. Therefore an orthonormal basis is $\{\langle 0, -1, 0, 0\rangle, \frac{1}{\sqrt{10}}\langle 3, 0, 1, 0\rangle, \frac{1}{\sqrt{110}}\langle \frac{1}{10}, 0, -\frac{3}{10}, 1\rangle\}$.

5. Let A be an invertible $n \times n$ matrix, and let c be a nonzero real number. Prove or disprove: $(cA)^{-1} = \frac{1}{c}A^{-1}$.

   Solution. $(cA)\left(\frac{1}{c}A^{-1}\right) = c \cdot \frac{1}{c}AA^{-1} = I_n$. Therefore $(cA)^{-1} = \frac{1}{c}A^{-1}$.

6. Let $A_{m\times n}$ and $B_{n\times p}$ be matrices. Prove that $(AB)^T = B^T A^T$, where A^T denotes the transpose of the matrix A.

   Solution. Let $A = [a_{ij}]$ and $B = [b_{ij}]$. The (i, j)-entry of $(AB)^T$ is the (j, i)-entry of AB, which is [jth row of A] $\cdot$ [ith column of B] $= a_{j1}b_{1i} + \ldots + a_{jn}b_{ni} = \displaystyle\sum_{k=1}^{n} a_{jk}b_{ki}$.
   On the other hand, the (i, j)-entry of $B^T A^T$ is [ith row of B^T] $\cdot$ [jth column of A^T] $=$ $b_{1i}a_{j1} + \ldots + b_{ni}a_{jn} = \displaystyle\sum_{k=1}^{n} a_{jk}b_{ki}$.
   Since the corresponding entries are equal, we have that $(AB)^T = B^T A^T$.

7. Let $A = \begin{bmatrix} 0 & 0 & -2 \\ 0 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$. Find the eigenvalues and corresponding eigenvector(s) of A.

Solution. The characteristic polynomial is

$$\begin{aligned}p(\lambda) &= \lambda\left((\lambda+1)(\lambda-1)\right) - 2(\lambda+1)\\ &= (\lambda+1)\left(\lambda(\lambda-1)-2\right)\\ &= (\lambda+1)(\lambda^2-\lambda-2)\\ &= (\lambda+1)^2(\lambda-2);\end{aligned}$$

therefore the eigenvalues are $\lambda_1 = -1$, $\lambda_2 = 2$.

For λ_1 we have the system $(-I - A)\mathbf{x} = \mathbf{0}$, which gives

$$\left[\begin{array}{rrr|r} -1 & 0 & 2 & 0\\ 0 & 0 & -2 & 0\\ 1 & 0 & -2 & 0\end{array}\right]$$

which reduces to

$$\left[\begin{array}{rrr|r} 1 & 0 & -2 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{array}\right].$$

A basis for the solution space is $\left\{\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$. So an eigenvector for λ_1 is $\mathbf{v}_1 = \begin{bmatrix}0\\1\\0\end{bmatrix}$.

Similarly for λ_2 we have the system $(2I - A)\mathbf{x} = \mathbf{0}$, which gives

$$\left[\begin{array}{rrr|r} 2 & 0 & 2 & 0\\ 0 & 3 & -2 & 0\\ 1 & 0 & 1 & 0\end{array}\right]$$

which reduces to

$$\left[\begin{array}{rrr|r} 1 & 0 & 1 & 0\\ 0 & 3 & -2 & 0\\ 0 & 0 & 0 & 0\end{array}\right].$$

A basis for the solution space is $\left\{\begin{bmatrix}-1\\ \frac{2}{3}\\ 1\end{bmatrix}\right\}$. So an eigenvector for λ_2 is $\mathbf{v}_2 = \begin{bmatrix}-1\\ \frac{2}{3}\\ 1\end{bmatrix}$.

8. Let A be an invertible $n \times n$ matrix. Prove that $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

Solution. We have $1 = \det(I_n) = \det(AA^{-1}) = \det(A)\det(A^{-1})$. Therefore $\det(A^{-1}) = \dfrac{1}{\det(A)}$.