Mathematics Department Qualifying Exam Spring 2007

Subject: Algebra

Part A. Do **five** of the following eight problems.

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1. Show that every subgroup of a cyclic group is cyclic.

2. Let G be a group of symmetries of the square with vertices A, B, C, D (that is, the group of rigid motions of the plane which transform the square into itself). Let $H_1 \subset G$ be the subgroup of G consisting of those symmetries which do not move the point A and $H_2 \subset G$ the subgroup of G consisting of rotations. Is H_1 a normal subgroup of G? Is H_2 a normal subgroup of G? Justify your answer.

3. Prove that all groups of order 4 are abelian.

4. Let R be the set of 2×2-matrices with real entries :

$$R = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

Then R forms a ring under matrix addition and multiplication. Let

$$S = \left\{ \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \in R \right\}$$

   (a) Prove that S is a subring of R.

   (b) Is S an ideal of R?

5. Consider the map $\theta : \mathbb{R}[x] \to \mathbb{R}$ given by $f(x) \mapsto f'(1)$, where f' is the derivative of f.

   (a) Is θ a group homomorphism from $(\mathbb{R}[x], +)$ to $(\mathbb{R}, +)$?

   (b) Is θ a ring homomorphism from $(\mathbb{R}[x], +, \cdot)$ to $(\mathbb{R}, +, \cdot)$?

6. Give an example of a finite group G and an integer n such that n divides the order of G but G has no subgroup of order n. Explain why G has no such subgroup.

7. (a) Show that every field is an integral domain.

   (b) Give an example of an integral domain that is not a field. Explain.

8. Let $\mathbb{Z}_5[x]$ be the ring of polynomials over the finite field $\mathbb{Z}_5$.

   (a) Show that $f(x) = x^3 + 3x + 2$ is irreducible over $\mathbb{Z}_5$.

   (b) Express $g(x) = x^4 + 4$ as a product of irreducible polynomials in $\mathbb{Z}_5[x]$.

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Part B. Do **five** of the following eight problems.

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1. Suppose that $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ are vectors in $\mathbb{R}^n$.

   (a) If $\mathbf{y} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_k\mathbf{x}_k$ where $a_1 \neq 0$, show that $span\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\} = span\{\mathbf{y}, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$.

   (b) If $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ is independent, show that $\{\mathbf{x}_1, \mathbf{x}_1 + \mathbf{x}_2, \mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3, \ldots, \mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_k\}$ is also independent.

2. Let V be a vector space.

   (a) Show that if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a linearly independent set of vectors in V, then so is every non-empty subset of S.

   (b) Show that if $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a linearly dependent set of vectors in V and $\mathbf{v}_{r+1}, \ldots, \mathbf{v}_n$ are any vectors in V, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r, \mathbf{v}_{r+1}, \ldots, \mathbf{v}_n\}$ is also linearly dependent.

3. Show that $A = \begin{bmatrix} 1 & 3 \\ -3 & -5 \end{bmatrix}$ is not diagonalizable.

4. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by $T(x, y) = (x + ky, -y)$. Show that T is one-to-one for every real value of k and that $T^{-1} = T$.

5. Let $\mathbf{v}_1 = \langle 0, 1, 0 \rangle$, $\mathbf{v}_2 = \langle -\frac{4}{5}, 0, \frac{3}{5} \rangle$, and $\mathbf{v}_3 = \langle \frac{3}{5}, 0, \frac{4}{5} \rangle$.

   (a) Check that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthonormal basis for $\mathbb{R}^3$ with the Euclidean inner product.

   (b) Express the vector $\mathbf{u} = \langle 1, 1, 1 \rangle$ as a linear combination of the vectors in S and find the coordinates of $\mathbf{u}$ with respect to the basis S.

6. Find the characteristic polynomial, eigenvalues, and eigenvectors for $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & -3 & 0 \end{bmatrix}$.

7. Suppose that $L : \mathbb{R}^3 \to \mathbb{R}^2$ is given by $L(x, y, z) = (x + 1, y - z)$. Is L a linear transformation? Explain.

8. Compute the rank and nullity of $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}$.