Algebra Qualifying Exam, Solutions Spring 2007

Part A.

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1. Show that every subgroup of a cyclic group is cyclic.

   Solution. Let $G =< g >$, and H a subgroup of G. Let i be the least positive integer such that $g^i \in H$.

   Let $h \in H$, then $h = g^j$ for some $i \leq j$. Using the division algorithm we get that

   $$j = iq + r$$

   for some integers q, r such that $0 \leq r < i$.

   Note that $g^j = g^{iq+r} = g^{iq}g^r$, thus $(g^{iq})^{-1}g^j = g^r$, which forces $g^r \in H$ (as both $(g^{iq})^{-1}$ and g^j live in H). This yields a contradiction unless $r = 0$. It follows that i divides j, and thus $H =< g^i >$.

2. Let G be a group of symmetries of the square with vertices A, B, C, D (that is, the group of rigid motions of the plane which transform the square into itself). Let $H_1 \subset G$ be the subgroup of G consisting of those symmetries which do not move the point A and $H_2 \subset G$ the subgroup of G consisting of rotations. Is H_1 a normal subgroup of G? Is H_2 a normal subgroup of G? Justify your answer.

   Solution. The group of symmetries of the square is called D_4, its order is 8.

   The group of all rotations, H_2, is cyclic of order 4. Since its index is two in D_4, then it is normal in D_4.

   Note that H_1 can be considered as the group generated by a reflection (that fixes A), thus $H_1 \cap H_2 = \{e\}$ (otherwise a rotation would fix a vertex). It follows that $D_4 = H_1H_2$. So, if H_1 were normal, then D_4 would be Abelian. However, D_4 is not Abelian, as a clockwise rotation in 90 degrees followed by a reflection with diagonal axis ℓ is not the same as a reflection with axis ℓ followed by a clockwise rotation in 90 degrees.

3. Prove that all groups of order 4 are abelian.

   Solution. Let $G = \{e, a, b, c\}$ be a group of order 4, and let $g \in G$.

   If the order of some nonidentity element in G is 4, then G is cyclic of order four and thus Abelian.

   If all nonidentity elements in G have order two. Note that ab cannot be a or b (that would force the existence of a second identity), thus $ab = c$, similarly $ba = c$. The same argument shows that $ac = ca = b$ and that $bc = cb = a$. Hence, G is abelian.

4. Let R be the set of 2×2-matrices with real entries :

$$R = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

Then R forms a ring under matrix addition and multiplication. Let

$$S = \left\{ \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \in R \right\}$$

(a) Prove that S is a subring of R.

(b) Is S an ideal of R?

Solution.

(a) Let $M, N \in S$, then the difference $M - N$ is clearly in S. Also, since the zero matrix is in S, then the only thing left to prove is that $MN \in S$. But this follows from

$$\begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \begin{bmatrix} x & 0 \\ y & z \end{bmatrix} = \begin{bmatrix} ax & 0 \\ cx + dy & dz \end{bmatrix} \in S$$

(b) S is not an ideal of R because, for example, using that the identity is in S we get

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \notin S$$

5. Consider the map $\theta : \mathbb{R}[x] \to \mathbb{R}$ given by $f(x) \mapsto f'(1)$, where f' is the derivative of f.

(a) Is θ a group homomorphism from $(\mathbb{R}[x], +)$ to $(\mathbb{R}, +)$?

(b) Is θ a ring homomorphism from $(\mathbb{R}[x], +, \cdot)$ to $(\mathbb{R}, +, \cdot)$?

Solution.

(a) We need to check if θ preserves the **additive** group operation of $\mathbb{R}[x]$.

$$\theta(p(x) + q(x)) = p'(1) + q'(1) = \theta(p(x)) + \theta(q(x))$$

So, θ is a group homomorphism from $(\mathbb{R}[x], +)$ to $(\mathbb{R}, +)$

(b) θ is not a ring homomorphism, as for $p(x) = x - 1$ and $q(x) = x^2$ we get

$$\theta(p(x)q(x)) = p'(1)q(1) + p(1)q'(1) = 1$$

and

$$\theta(p(x))\theta(q(x)) = p'(1)q'(1) = 2$$

6. Give an example of a finite group G and an integer n such that n divides the order of G but G has no subgroup of order n. Explain why G has no such subgroup.

Solution. We know that A_5 is simple, and that it has order 60. If A_5 had a subgroup of order 30, then that subgroup would have index two and, thus, it would be normal. It follows that A_5 has no subgroup of order 30.

7. (a) Show that every field is an integral domain.
   (b) Give an example of an integral domain that is not a field. Explain.

Solution.

(a) A field is clearly a commutative ring.
Let $ab = 0$, with $a \neq 0$. Since in a field every nonzero element has an inverse, then $ab = 0$ implies $b = a^{-1}0 = 0$, so $b = 0$. It follows that every field has no zero divisors.

(b) $\mathbb{Z}$ has no zero divisors and, for example, 2 has no inverse in $\mathbb{Z}$

8. Let $\mathbb{Z}_5[x]$ be the ring of polynomials over the finite field $\mathbb{Z}_5$.
   (a) Show that $f(x) = x^3 + 3x + 2$ is irreducible over $\mathbb{Z}_5$.
   (b) Express $g(x) = x^4 + 4$ as a product of irreducible polynomials in $\mathbb{Z}_5[x]$.

Solution.

(a) Since the degree of $f(x)$ is three, then for it to be irreducible over $\mathbb{Z}_5$ it is enough to check that it has no zeros (roots) in $\mathbb{Z}_5$. This is easy to check as (all computations in $\mathbb{Z}_5$!)

$$f(0) = 2 \qquad f(1) = 1 \qquad f(2) = 1 \qquad f(3) = 3 \qquad f(4) = 3$$

(b) Since $4 \equiv -1 \pmod 5$ then $g(x) = x^4 - 1 \in \mathbb{Z}_5[x]$. Factoring as usual we get

$$g(x) = (x^2 - 1)(x^2 + 1) = (x^2 - 1)(x^2 - 4) = (x - 1)(x + 1)(x - 2)(x + 2)$$

Since all the factors are linear we are done.

Part B.

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1. Suppose that $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ are vectors in $\mathbb{R}^n$.

   (a) If $\mathbf{y} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_k\mathbf{x}_k$ where $a_1 \neq 0$, show that $span\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\} = span\{\mathbf{y}, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$.

   (b) If $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ is independent, show that $\{\mathbf{x}_1, \mathbf{x}_1+\mathbf{x}_2, \mathbf{x}_1+\mathbf{x}_2+\mathbf{x}_3, \ldots, \mathbf{x}_1+\mathbf{x}_2+\cdots+\mathbf{x}_k\}$ is also independent.

Solution.

(a) Note that $\mathbf{y} \in span\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$, then we clearly have

$$span\{\mathbf{y}, \mathbf{x}_2, \ldots, \mathbf{x}_k\} \subset span\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$$

Now let

$$\mathbf{v} = \alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \cdots + \alpha_k\mathbf{x}_k \in span\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$$

then

$$\begin{aligned}
\mathbf{v} &= \frac{\alpha_1}{a_1}(a_1\mathbf{x}_1) + \alpha_2\mathbf{x}_2 + \cdots + \alpha_k\mathbf{x}_k \\
&= \frac{\alpha_1}{a_1}(a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_k\mathbf{x}_k) + \alpha_2\mathbf{x}_2 + \cdots + \alpha_k\mathbf{x}_k - \frac{\alpha_1}{a_1}(a_2\mathbf{x}_2 + \cdots + a_k\mathbf{x}_k) \\
&= \frac{\alpha_1}{a_1}\mathbf{y} + \left(\alpha_2 - \frac{\alpha_1}{a_1}\right)\mathbf{x}_2 + \cdots + \left(\alpha_k - \frac{\alpha_1}{a_1}\right)\mathbf{x}_k
\end{aligned}$$

which is an element of $span\{\mathbf{y}, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$. So,

$$span\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\} \subset span\{\mathbf{y}, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$$

(b) Let

$$0 = \alpha_1\mathbf{x}_1 + \alpha_2(\mathbf{x}_1+\mathbf{x}_2) + \alpha_3(\mathbf{x}_1+\mathbf{x}_2+\mathbf{x}_3) + \cdots + \alpha_k(\mathbf{x}_1+\mathbf{x}_2+\cdots+\mathbf{x}_k)$$

be a linear combination of the elements of the set $\{\mathbf{x}_1, \mathbf{x}_1+\mathbf{x}_2, \mathbf{x}_1+\mathbf{x}_2+\mathbf{x}_3, \ldots, \mathbf{x}_1+\mathbf{x}_2+\cdots+\mathbf{x}_k\}$.
We re-write the previous equation as

$$0 = (\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_k)\mathbf{x}_1 + (\alpha_2+\alpha_3+\cdots+\alpha_k)\mathbf{x}_2 + \cdots + (\alpha_{k-1}+\alpha_k)\mathbf{x}_{k-1} + \alpha_k\mathbf{x}_k$$

which is a linear combination of the elements of the set $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$. Since this set is linearly independent, then

$$0 = \alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_k \qquad 0 = \alpha_2+\alpha_3+\cdots+\alpha_k \qquad 0 = \alpha_{k-1}+\alpha_k \qquad 0 = \alpha_k$$

But, $0 = \alpha_k$ plugged into $0 = \alpha_{k-1} + \alpha_k$ yields $0 = \alpha_{k-1}$, and then we can repeat this process as many times as necessary to get that all the α_i's have to be zero! This implies that the set $\{\mathbf{x}_1, \mathbf{x}_1+\mathbf{x}_2, \mathbf{x}_1+\mathbf{x}_2+\mathbf{x}_3, \ldots, \mathbf{x}_1+\mathbf{x}_2+\cdots+\mathbf{x}_k\}$ is linearly independent.

2. Let V be a vector space.

   (a) Show that if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a linearly independent set of vectors in V, then so is every non-empty subset of S.

   (b) Show that if $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a linearly dependent set of vectors in V and $\mathbf{v}_{r+1}, \ldots, \mathbf{v}_n$ are any vectors in V, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r, \mathbf{v}_{r+1}, \ldots, \mathbf{v}_n\}$ is also linearly dependent.

   Solution.

   (a) If a non-empty subset T of S were linearly dependent, then there would be scalars (not all zero) that could be used to write a linear combination of the elements in T that is equal to zero. So, now we could extend this linear combination to a linear combination of the elements in S by just multiplying by zero the elements in $S \setminus T$. This would yield a non-trivial linear combination of the elements of S that is equal to zero. A contradiction.

   (b) Using the same idea used in part (a). If there is a non-trivial linear combination of the elements in $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ that is equal to zero, then we extend it to the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r, \mathbf{v}_{r+1}, \ldots, \mathbf{v}_n\}$ by just multiplying the elements $\mathbf{v}_{r+1}, \ldots, \mathbf{v}_n$ by zero.

3. Show that $A = \begin{bmatrix} 1 & 3 \\ -3 & -5 \end{bmatrix}$ is not diagonalizable.

   Solution. The characteristic polynomial of A is

$$\begin{aligned} \chi_A(\lambda) &= \begin{vmatrix} 1-\lambda & 3 \\ -3 & -5-\lambda \end{vmatrix} \\ &= (1-\lambda)(-5-\lambda)+9 \\ &= \lambda^2 + 4\lambda + 4 \\ &= (\lambda+2)^2 \end{aligned}$$

   We now look at the eigenspace associated to $\lambda = -2$. We need to solve the equation $Av = -2v$, which yields the system of equations

$$x + 3y = -2x \qquad\qquad -3x - 5y = -2y$$

   which has solution space spanned by $(1, -1)$.

   It follows that $\lambda = -2$ has geometric multiplicity one and algebraic multiplicity two. Hence, A is not diagonalizable.

4. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by $T(x, y) = (x + ky, -y)$. Show that T is one-to-one for every real value of k and that $T^{-1} = T$.

Solution. Let us look at the kernel of T.

$$\begin{aligned} Ker(T) &= \{(x,y)\in\mathbb{R}^2;\ T(x,y)=(0,0)\} \\ &= \{(x,y)\in\mathbb{R}^2;\ (x+ky,-y)=(0,0)\} \\ &= \{(x,y)\in\mathbb{R}^2;\ x+ky=0 \text{ and } -y=0\} \\ &= \{(0,0)\} \end{aligned}$$

So, T is one-to-one.

It is easy to see that $T\circ T(x,y)=(x,y)$ for all $(x,y)\in\mathbb{R}^2$

5. Let $\mathbf{v}_1=\langle 0,1,0\rangle$, $\mathbf{v}_2=\langle -\frac{4}{5},0,\frac{3}{5}\rangle$, and $\mathbf{v}_3=\langle \frac{3}{5},0,\frac{4}{5}\rangle$.
   (a) Check that $S=\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ is an orthonormal basis for $\mathbb{R}^3$ with the Euclidean inner product.
   (b) Express the vector $\mathbf{u}=\langle 1,1,1\rangle$ as a linear combination of the vectors in S and find the coordinates of $\mathbf{u}$ with respect to the basis S.

Solution.

(a) It is easy to check that the norm of $\mathbf{v}_1,\mathbf{v}_2$ and $\mathbf{v}_3$ is one. Also,

$$<\mathbf{v}_1,\mathbf{v}_2,>=0 \qquad <\mathbf{v}_1,\mathbf{v}_3>=0 \qquad <\mathbf{v}_2,\mathbf{v}_3>=-\frac{12}{25}+\frac{12}{25}=0$$

Since a set of pairwise orthogonal vectors must be linearly independent, then S is a set having three linearly independent vectors in a three-dimensional vector space, thus S is a basis.

(b) We want to find α,β and γ such that

$$\mathbf{u}=\alpha\mathbf{v}_1+\beta\mathbf{v}_2+\gamma\mathbf{v}_3$$

Since $\mathbf{v}_2$ and $\mathbf{v}_3$ have a zero in their second component, then $\alpha=1$. So, we just need to solve the system

$$-\frac{4}{5}\beta+\frac{3}{5}\gamma=1 \qquad \frac{3}{5}\beta+\frac{4}{5}\gamma=1$$

It follows that $\beta=-\frac{1}{5}$ and $\gamma==\frac{7}{5}$. So,

$$\mathbf{u}=\left(1,-\frac{1}{5},\frac{7}{5}\right)_S$$

Express the vector $\mathbf{u}=\langle 1,1,1\rangle$ as a linear combination of the vectors in S and find the coordinates of $\mathbf{u}$ with respect to the basis S.

6. Find the characteristic polynomial, eigenvalues, and eigenvectors for $A=\begin{bmatrix}1&1&1\\0&2&-1\\0&-3&0\end{bmatrix}$.

Solution. We first find the characteristic polynomial of A

$$\begin{aligned} \chi_A(\lambda) &= \begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & -1 \\ 0 & -3 & -\lambda \end{vmatrix} \\ &= (1-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ -3 & -\lambda \end{vmatrix} \\ &= (1-\lambda)(\lambda^2-2\lambda-3) \\ &= (1-\lambda)(\lambda-3)(\lambda+1) \end{aligned}$$

Since all the eigenvalues of A are distinct (they are $\lambda = -1, 1,$ and 3). Then A is diagonalizable to

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Now we look for eigenvectors. First of all, just by looking at A we realize that $A\mathbf{e}_1 = \mathbf{e}_1$, so that would be an eigenvector associated to the eigenvalue $\lambda = 1$.

For $\lambda = -1$ we need to solve the equation $Av = -v$, which yields the system of equations

$$x+y+z=-x \qquad 2y-z=-y \qquad -3y=-z$$

which has solution subspace spanned by $(-2, 1, 3)$. Thus we can consider $(-2, 1, 3)$ as an eigenvector for $\lambda = -1$.

For $\lambda = 3$ we need to solve the equation $Av = 3v$, which yields the system of equations

$$x+y+z=3x \qquad 2y-z=3y \qquad -3y=3z$$

which has solution subspace spanned by $(0, 1, -1)$. Thus we can consider $(0, 1, -1)$ as an eigenvector for $\lambda = 3$.

7. Suppose that $L : \mathbb{R}^3 \to \mathbb{R}^2$ is given by $L(x, y, z) = (x+1, y-z)$. Is L a linear transformation? Explain.

Solution. It is not. If this function were linear, then its kernel would be a subspace. However, since $L(0, 0, 0) = (1, 0)$, then the zero vector is not in the 'kernel' of L. This is a contradiction.

8. Compute the rank and nullity of $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}$.

Solution. Since the first two columns are linearly independent, then the rank is at least two. Now note that the sum of the first and second column yield the third column plus the vector $\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$. This implies that the third column is not in the span of the first two (the two zeros in the 'extra vector' do the work). It follows that the rank is three, and thus the nullity is zero.