Mathematics Department Qualifying Exam : Algebra Spring 2008

Part A. Solve **five** of the following eight problems :

1. Let $\theta : \mathbb{Z} \to S_5$ be a group homomorphism such that $\theta(1) = (123)(45)$. Find $\theta(-4)$ and $\operatorname{Ker}\theta$.

2. Let G be the set of all real-valued functions defined on $\mathbb{R}$. Then $(G, +)$ is a group where $+$ stands for the usual addition of functions. Put $N = \{f \in G \mid f(2008) = 0\}$. Prove that $N \trianglelefteq G$ and $G/N \cong \mathbb{R}$.

3. TRUE/FALSE : $S_6 \cong \mathbb{S}_3 \oplus S_5$. Prove your answer!

4. Let G be a finite group and k a natural number that is relatively prime to $|G|$. Prove that the map $\theta : G \to G : x \to x^k$ is a bijection.

5. Let G be a finite group, $N \trianglelefteq G$ and $H \leq G$ such that $|H|$ and $[G : N]$ are relatively prime. Prove that $H \leq N$.

6. Let $G = \mathbb{R} \setminus \{-1\}$. We define a binary operation $*$ on G as follows :

$$a * b = ab + a + b \quad \text{for all } a, b \in G$$

It is given that G is a group under this operation.

(a) What is the identity element in G?

(b) What is the inverse of $a \in G$?

(c) Solve for x : $2 * x * 3 = 7$

7. Let R be a ring such that $a^2 = a$ for all $a \in R$.

(a) Prove that $a + a = 0$ for all $a \in R$ (hint : consider $(a + a)^2$).

(b) Prove that R is commutative (hint : consider $(a + b)^2$).

8. Let R be the set of all polynomials with real coefficients. Then $(R, +, \cdot)$ is a ring under the usual addition and multiplication of polynomials. Put $S = \{p(x) \in R \mid p'(0) = 0\}$ where $p'(x)$ is the usual derivative of $p(x)$ with respect to x.

(a) Prove that S is a subring of R.

(b) Is S an ideal of R?

Part B is on the back!!!

Part B. Solve **five** of the following eight problems :

1. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be linearly independent vectors in $\mathbb{R}^n$ and A a non-singular $n \times n$ matrix. Prove that $A\mathbf{v}_1, A\mathbf{v}_2, \ldots, A\mathbf{v}_k$ are linearly independent.

2. Let

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \mathbf{x}_4 = \begin{pmatrix} 2 \\ 7 \\ 4 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_5 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Find a subset of $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5\}$ that is a basis of $\operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5\}$.

3. Let A and B be similar $n \times n$ matrices. Prove that there exist $n \times n$ matrices S and T such that S is non-singular, $A = ST$ and $B = TS$.

4. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that $T\begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $T\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. Find $T\begin{pmatrix} x \\ y \end{pmatrix}$ for all $x, y \in \mathbb{R}$.

5. Let V be the set of all polynomials of degree at most three. For $f(x), g(x) \in V$, we define the inner product of $f(x)$ and $g(x)$ as

$$\langle f(x), g(x) \rangle = \int_0^1 f(x)g(x)\,dx$$

Find a basis for the subspace W of V of all elements in V that are orthogonal to $1 - x$.

6. Is the matrix $\begin{bmatrix} 1 & 0 & 0 \\ -2 & -1 & -3 \\ 2 & 2 & 4 \end{bmatrix}$ diagonalizable? Justify your answer!

7. Let A be an $n \times n$ matrix, λ, μ two different eigenvalues of A, $\mathbf{x}$ an eigenvector for A corresponding to the eigenvalue λ and $\mathbf{y}$ an eigenvector for A^T corresponding to the eigenvalue μ. Prove that $\mathbf{x}$ and $\mathbf{y}$ are orthogonal.

8. Let n be odd and A an $n \times n$ matrix whose entries are real numbers. Prove that $A^2 + I \neq O$ where I is the $n \times n$ identity matrix and O is the $n \times n$ zero matrix (hint : determinants).