Algebra Qualifying Exam, Solutions

Spring 2008

Part A.

1. Let $\theta : \mathbb{Z} \to S_5$ be a group homomorphism such that $\theta(1) = (123)(45)$. Find $\theta(-4)$ and $\operatorname{Ker}\theta$.

Solution. θ is a homomorphism going from an additive group into a multiplicative group. Thus the homomorphism properties look like

$$\theta(n+m) = \theta(n)\theta(m) \qquad \text{and} \qquad \theta(-n) = \theta(n)^{-1}$$

So, as $\mathbb{Z}$ is cyclic with generator 1, and we know what $\theta(1)$ is, then

$$\theta(n) = \theta(1)^n = [(123)(45)]^n = (123)^n(45)^n$$

where the last step is obtained using that (123) and (45) are disjoint. Then,

$$\theta(-4) = (123)^{-4}(45)^{-4} = (321)^4(45)^4 = (321)$$

$$\begin{aligned} Ker(\theta) &= \{n \in \mathbb{Z};\ \theta(n) = e\} \\ &= \{n \in \mathbb{Z};\ [(123)(45)]^n = e\} \\ &= \{n \in \mathbb{Z};\ n \text{ is divisible by the order of } (123)(45)\} \\ &= \{n \in \mathbb{Z};\ 6 \text{ divides } n\} \quad \to (\text{the order of } (123)(45) \text{ is } 3 \cdot 2 = 6) \\ &= 6\mathbb{Z} \end{aligned}$$

2. Let G be the set of all real-valued functions defined on $\mathbb{R}$. Then $(G, +)$ is a group where $+$ stands for the usual addition of functions. Put $N = \{f \in G \mid f(2008) = 0\}$. Prove that $N \trianglelefteq G$ and $G/N \cong \mathbb{R}$.

Solution. Consider the map $\phi : G \to \mathbb{R}$ defined by $\phi(f) = f(2008)$.
Let $f, g \in G$, then

$$\phi(f+g) = (f+g)(2008) = f(2008) + g(2008) = \phi(f) + \phi(g)$$

So, ϕ is a homomorphism of (additive) groups.
For any given $y \in \mathbb{R}$ consider the map $f_y : \mathbb{R} \to \mathbb{R}$ defined by $f_y(x) = y$ for all $x \in \mathbb{R}$. Since, $f_y \in G$ and $f_y(2008) = y$, then ϕ is onto.
It is easy to see that $Ker(\phi) = N$. Hence $N \trianglelefteq G$ and, by the first isomorphism theorem, $G/N \cong \mathbb{R}$.

3. TRUE/FALSE : $S_6 \cong \mathbb{S}_3 \oplus S_5$. Prove your answer!

Solution. False. Note that the element $((123),(12345)) \in S_3 \oplus S_5$ has order 15. On the other hand, S_6 does not have elements of order 15.

4. Let G be a finite group and k a natural number that is relatively prime to $|G|$. Prove that the map $\theta : G \to G : x \to x^k$ is a bijection.

Solution. Since $|G| = n < \infty$, then we just need to show that θ is onto.
We know that there are $\alpha, \beta \in \mathbb{Z}$ such that $1 = \alpha k + \beta n$, then

$$x = x^{\alpha k+\beta n} = x^{\alpha k}x^{\beta n} = x^{\alpha k}(x^n)^{\beta} = x^{\alpha k}$$

for all $x \in G$. So, $\theta(x^{\alpha}) = x$.

5. Let G be a finite group, $N \trianglelefteq G$ and $H \leq G$ such that $|H|$ and $[G : N]$ are relatively prime. Prove that $H \leq N$.

Solution. Let $h \in H$, then the order of $hN \in G/N$ divides the order of h. It follows that the order of hN is a common divisor of $[G : N]$ and $|H|$. Hence, the order of hN is one, and thus $h \in N$.

6. Let $G = \mathbb{R} \setminus \{-1\}$. We define a binary operation $*$ on G as follows :

$$a * b = ab + a + b \quad \text{for all } a, b \in G$$

It is given that G is a group under this operation.

(a) What is the identity element in G?

(b) What is the inverse of $a \in G$?

(c) Solve for x : $2 * x * 3 = 7$

Solution. First notice that the operation is commutative.

(a) We want to find e such that $a = a * e = ae + a + e$ for all $a \in G$. It follows that

$$0 = ae + e = e(a - 1)$$

for all $a \in G$. It follows that $e = 0$.

(b) We look for b such that $0 = e = a * b = ab + a + b$. We solve for b and we get

$$b = \frac{-a}{a+1}$$

Note that we are using that $a + 1 \neq 0$, for all $a \in G$

(c) Since $2 * 3 = 2 \cdot 3 + 2 + 3 = 11$, then the equation becomes

$$11 * x = 7$$

The inverse of 11 is

$$11^{-1} = \frac{-11}{12}$$

So,

$$x = \frac{-11}{12} * 7 = \frac{-11}{12} \cdot 7 + \frac{-11}{12} + 7 = \frac{-22}{3} + 7 = -\frac{1}{3}$$

7. Let R be a ring such that $a^2 = a$ for all $a \in R$.

(a) Prove that $a + a = 0$ for all $a \in R$ (hint : consider $(a + a)^2$).

(b) Prove that R is commutative (hint : consider $(a + b)^2$).

Solution.

(a) Using that $a^2 = a$ for all $a \in R$ we get

$$a + a = (a + a)^2 = a^2 + a + a + a^2 = 2(a + a)$$

So, $a + a = 0$, or $a = -a$.

(b) We now consider

$$a + b = (a + b)^2 = a^2 + ab + ba + b^2 = a + ab + ba + b$$

So, $0 = ab + ba$. This together with part (a) imply $ab = ba$ for all $a, b \in R$

8. Let R be the set of all polynomials with real coefficients. Then $(R, +, \cdot)$ is a ring under the usual addition and multiplication of polynomials. Put $S = \{p(x) \in R \,|\, p'(0) = 0\}$ where $p'(x)$ is the usual derivative of $p(x)$ with respect to x.

(a) Prove that S is a subring of R.

(b) Is S an ideal of R?

Solution.

(a) Let $p, q \in S$, then

$$(p-q)'(0) = p'(0) - q'(0) = 0$$

and

$$(pq)'(0) = p'(0)q(0) + p(0)q'(0) = 0$$

So, both closure laws and the existence of additive inverses hold in S. It is clear that S is nonempty, as the zero polynomial is in S.

(b) Let $p(x) = x^2 + 1 \in S$ and $q(x) = x \in R$ we get that $p(x)q(x) = x^3 + x$ and $(pq)'(x) = 3x^2 + 1$ which is nonzero at $x = 0$.

So, S is not an ideal of R.

Part B.

1. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be linearly independent vectors in $\mathbb{R}^n$ and A a non-singular $n \times n$ matrix. Prove that $A\mathbf{v}_1, A\mathbf{v}_2, \ldots, A\mathbf{v}_k$ are linearly independent.

Solution. Let $\alpha_1, \alpha_2, \cdots, \alpha_k \in \mathbb{R}$ and consider

$$\begin{aligned} 0 &= \alpha_1 A\mathbf{v}_1 + \alpha_2 A\mathbf{v}_2 + \cdots + \alpha_k A\mathbf{v}_k \\ &= A(\alpha_1\mathbf{v}_1) + A(\alpha_2\mathbf{v}_2) + \cdots + A(\alpha_k\mathbf{v}_k) \\ &= A(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k) \end{aligned}$$

which forces

$$0 = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k$$

because A is non-singular. But, since the $\mathbf{v}_i$'s are linearly independent, then all the α_i's are zero.

2. Let

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \ \mathbf{x}_2 = \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}, \ \mathbf{x}_3 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \mathbf{x}_4 = \begin{pmatrix} 2 \\ 7 \\ 4 \end{pmatrix} \ \text{and} \ \mathbf{x}_5 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Find a subset of $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5\}$ that is a basis of $\text{span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5\}$.

Solution. Call S the space spanned by the $\mathbf{x}_i$'s.
Note that $\mathbf{x}_4 - \mathbf{x}_2 = 2\mathbf{e}_2$. Since we are in $\mathbb{R}$, then we have $\mathbf{e}_2 \in S$. Subtracting this from $\mathbf{x}_5$ we obtain $\mathbf{e}_1 \in S$. Now that we have $\mathbf{e}_1$ and $\mathbf{e}_2$ we obtain $\mathbf{e}_3$ from either $\mathbf{x}_4$ or $\mathbf{x}_2$. It follows that S contains the canonical basis of $\mathbb{R}^3$, and thus $S = \mathbb{R}^3$. Moreover, we obtained the canonical basis by only playing with $\mathbf{x}_2, \mathbf{x}_4$, and $\mathbf{x}_5$, then these three vectors define a basis of S (three generators in a space of dimension three).

3. Let A and B be similar $n \times n$ matrices. Prove that there exist $n \times n$ matrices S and T such that S is non-singular, $A = ST$ and $B = TS$.

Solution. Since A is similar to B, then there is an invertible matrix P such that $A = PBP^{-1}$.
Let $S = P$ (invertible) and $T = BP^{-1}$, then $A = ST$ and $B = TS$.

4. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that $T\begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $T\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. Find $T\begin{pmatrix} x \\ y \end{pmatrix}$ for all $x, y \in \mathbb{R}$.

Solution. We need to write any vector $v = \begin{pmatrix} x \\ y \end{pmatrix}$ as a linear combination of $v_1 = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.
So we set $v = \alpha v_1 + \beta v_2$, which yields the system of equations (the unknowns are α and β)

$$x = 5\alpha + 3\beta \qquad\qquad y = 3\alpha + 2\beta$$

It follows that $\alpha = 2x - 3y$ and $\beta = 5y - 3x$. Hence,

$$\begin{aligned} T\begin{pmatrix} x \\ y \end{pmatrix} &= T\left[(2x-3y)\begin{pmatrix} 5 \\ 3 \end{pmatrix} + (5y-3x)\begin{pmatrix} 3 \\ 2 \end{pmatrix}\right] \\ &= (2x-3y)T\begin{pmatrix} 5 \\ 3 \end{pmatrix} + (5y-3x)T\begin{pmatrix} 3 \\ 2 \end{pmatrix} \\ &= (2x-3y)\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + (5y-3x)\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -x+2y \\ 4x-6y \\ 3x-4y \end{pmatrix} \end{aligned}$$

5. Let V be the set of all polynomials of degree at most three. For $f(x), g(x) \in V$, we define the inner product of $f(x)$ and $g(x)$ as

$$\langle f(x), g(x) \rangle = \int_0^1 f(x)g(x)\,dx$$

Find a basis for the subspace W of V of all elements in V that are orthogonal to $1 - x$.

Solution. Let $p(x) = \delta x^3 + ax^2 + bx + c \in W$.

$$\begin{aligned}
\langle p(x), 1-x\rangle &= \int_0^1 p(x)(1-x)\,dx \\
&= \int_0^1 -\delta x^4 + (\delta - a)x^3 + (a-b)x^2 + (b-c)x + c\,dx \\
&= \left(-\frac{\delta}{5}x^5 + \frac{\delta - a}{4}x^4 + \frac{a-b}{3}x^3 + \frac{b-c}{2}x^2 + cx + K\right)\Bigg|_0^1 \\
&= -\frac{\delta}{5} + \frac{\delta - a}{4} + \frac{a-b}{3} + \frac{b-c}{2} + c
\end{aligned}$$

Since we want $p(x)$ to be orthogonal to $1-x$, then we are asking for

$$-\frac{\delta}{5} + \frac{\delta - a}{4} + \frac{a-b}{3} + \frac{b-c}{2} + c = 0$$

or

$$c = -\left(\frac{\delta}{10} + \frac{a}{6} + \frac{b}{3}\right)$$

Thus,

$$\begin{aligned}
p(x) &= \delta x^3 + ax^2 + bx - \left(\frac{\delta}{10} + \frac{a}{6} + \frac{b}{3}\right) \\
&= \delta\left(x^3 - \frac{1}{10}\right) + a\left(x^2 - \frac{1}{6}\right) + b\left(x - \frac{1}{3}\right)
\end{aligned}$$

Hence, a basis for W is

$$\left\{x^3 - \frac{1}{10},\ x^2 - \frac{1}{6},\ x - \frac{1}{3}\right\}$$

6. Is the matrix $\begin{bmatrix} 1 & 0 & 0 \\ -2 & -1 & -3 \\ 2 & 2 & 4 \end{bmatrix}$ diagonalizable? Justify your answer!

Solution. The characteristic polynomial of the matrix above (let's call it A) is

$$\begin{aligned}
\chi_A(\lambda) &= \begin{vmatrix} 1-\lambda & 0 & 0 \\ -2 & -1-\lambda & -3 \\ 2 & 2 & 4-\lambda \end{vmatrix} \\
&= (1-\lambda)\begin{vmatrix} -1-\lambda & -3 \\ 2 & 4-\lambda \end{vmatrix} \\
&= (1-\lambda)(\lambda^2 - 3\lambda - 10) \\
&= (1-\lambda)(\lambda - 5)(\lambda + 2)
\end{aligned}$$

Since all A's eigenvalues are distinct, then A is diagonalizable.

7. Let A be an $n \times n$ matrix, λ, μ two different eigenvalues of A, $\mathbf{x}$ an eigenvector for A corresponding to the eigenvalue λ and $\mathbf{y}$ an eigenvector for A^T corresponding to the eigenvalue μ. Prove that $\mathbf{x}$ and $\mathbf{y}$ are orthogonal.

Solution. We know that $A\mathbf{x} = \lambda\mathbf{x}$ (which implies $\mathbf{x}^T A^T = \lambda\mathbf{x}^T$) and that $A^T\mathbf{y} = \mu\mathbf{y}$.
Note that

$$\begin{aligned}\lambda(x \cdot y) &= \lambda(x^T y) \\ &= (\lambda x^T)y \\ &= (\mathbf{x}^T A^T)y \\ &= \mathbf{x}^T(A^T y) \\ &= \mathbf{x}^T(\mu y) \\ &= \mu(x^T y) \\ &= \mu(x \cdot y)\end{aligned}$$

Since $\lambda \neq \mu$, then $\lambda(x \cdot y) = \mu(x \cdot y)$ forces $x \cdot y = 0$.

8. Let n be odd and A an $n \times n$ matrix whose entries are real numbers. Prove that $A^2 + I \neq O$ where I is the $n \times n$ identity matrix and O is the $n \times n$ zero matrix (hint : determinants).

Solution. If $A^2 + I = O$, then $A^2 = -I$, and thus $\det(A^2) = \det(-I)$.
Since n is odd $\det(-I) = -1$, then we get $\det(A)^2 = -1$, which forces $\det(A) \notin \mathbb{R}$. This contradicts the fact that the entries of A are real.