
Part A. Solve **five** of the following eight problems:

1. Let $G = \{x \in \mathbb{R} \mid x > 0 \text{ and } x \neq 1\}$. Define the operation $*$ on G by $a * b = a^{\ln b}$, for all $a, b \in G$.

   (a) Prove that G is an abelian group under the operation $*$.

   (b) Show that G is isomorphic to the multiplicative group $\mathbb{R}^{\times}$.

2. Let G_1, G_2 be groups.

   (a) If $H_1 \leq G_1$ and $H_2 \leq G_2$ prove that $H_1 \times H_2 \leq G_1 \times G_2$.

   (b) TRUE/FALSE : If $H \leq G_1 \times G_2$ then $H = H_1 \times H_2$ for some $H_1 \leq G_1$ and some $H_2 \leq G_2$. Prove your answer!

3. If $\phi : S_3 \to \mathbb{Z}_3$ is a group homomorphism, show that $\phi(g) = 0$ for all $g \in S_3$.

4. Define $f : \mathbb{Z}_{mn} \to \mathbb{Z}_m \times \mathbb{Z}_n$ by $f([x]_{mn}) = ([x]_m, [x]_n)$. Show that f is well-defined, and that f is bijective if and only if $\gcd(m, n) = 1$.

5. Let $H \trianglelefteq G$ and for any $g \in G$ define $n_H(g)$ to be the least positive integer such that $g^{n_H(g)} \in H$. Show that $n_H(g)$ divides the order of g.

6. Let R be a commutative ring. An element $r \in R$ is called *nilpotent* if $r^n = 0$ for some integer $n > 0$. Prove that $a + b$ is nilpotent if a and b are nilpotent elements of R.

7. Show that the set of matrices $A \in M_n(\mathbb{R})$ such that $Av = 0$ for some fixed $v \in \mathbb{R}^n$ is a left ideal of $M_n(\mathbb{R})$.

8. Let

$$S = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \mid a, b \in \mathbb{R} \right\} \subset M_2(\mathbb{R})$$

   (a) Show that S is a subring of $M_2(\mathbb{R})$, the ring of 2×2 matrices with real entries.

   (b) Show that S and $\mathbb{C}$ are isomorphic rings.

Part B is on the back!!!

Part B. Solve **five** of the following eight problems :

1. Recall that $\mathcal{P}_5$ is the set containing the zero polynomial and all polynomials of degree at most five with real coefficients. Show that the derivative defines a linear transformation from $\mathcal{P}_5$ to itself. Is it onto? Find the matrix for this map in the standard basis.

2. Show that

$$U = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{R} \text{ and } a + b + c + d = 0 \right\} \subset M_2(\mathbb{R})$$

is a subspace of $M_2(\mathbb{R})$. Find a basis for U.

3. Find a basis for the orthogonal complement of the subspace $W = span\{(1, 2, -1, 0), (0, 1, 1, 3)\}$ of $\mathbb{R}^4$.

4. Let T be the linear transformation of $\mathbb{R}^3$ with standard matrix $\begin{bmatrix} 1 & 5 & 2 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$. Find the matrix of T with respect to the basis $\mathbb{B} = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$.

5. Let $F : \mathbb{R}^4 \to \mathbb{R}^3$ be any linear transformation such that

$$\operatorname{Ker} F = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \;\middle|\; \begin{array}{r} x_1 - 2x_2 + x_3 - x_4 = 0 \\ x_1 - x_2 - 2x_3 + x_4 = 0 \\ x_1 - 3x_2 + 4x_3 - 3x_4 = 0 \end{array} \right\}.$$

   (a) Find the dimension of Ker F and a basis for it.
   (b) Give an example of such a linear transformation F.
   (c) For the example you gave in (b), find a basis for the range of F.

6. A square matrix B is *skew-symmetric* if $B^T = -B$. Suppose that the square matrix A is skew-symmetric and invertible. Prove that A^{-1} is also skew-symmetric.

7. Diagonalize the following matrix

$$A = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

Then give a basis of $\mathbb{R}^3$ for which A 'becomes' diagonal.

8. Consider the subspace $U = \{(x, y, z) \in \mathbb{R}^3 | \, 2x - y - 3z = 0\} \subset \mathbb{R}^3$ and the set of vectors $S = \{(1, -1, 1), (4, 2, 2)\} \subset \mathbb{R}^3$.

   (a) Complete S to a basis in $\mathbb{R}^3$.
   (b) Show that $U = span(S)$.