
Part A.

1. Let $G = \{x \in \mathbb{R} \mid x > 0 \text{ and } x \neq 1\}$. Define the operation $*$ on G by $a * b = a^{\ln b}$, for all $a, b \in G$.

1. Prove that G is an abelian group under the operation $*$.
2. Show that G is isomorphic to the multiplicative group $\mathbb{R}^{\times}$.

Solution.

1. First note that the product of any two elements in G is well defined, as $x \mapsto \ln x$ is a well-defined function.

   Since $a^{\ln b} = 1$ only when $a = 1$ or $\ln b = 0$ (which forces $b = 1$), and $1 \notin G$, then closure of $*$ holds.

   The identity must be an element a (using e will get us into notation problems, as $\ln x = \log_e x$) such that $b = a * b = a^{\ln b}$. If we apply $\ln$ both sides we get

   $$\ln b = \ln a^{\ln b} = a^{\ln b} \ln a$$

   thus $\ln a = 1$, and thus $a = e$. I guess that notation problem is solved by now.

   It is clear that $b = b * e = b^{\ln e}$ for all $b \in G$. Thus e is the identity of G.

   The inverse of $b \in G$ is found by solving $e = a * b = a^{\ln b}$, which after applying $\ln$ both sides yields $1 = \ln a \ln b$. This equation can always be solved in G, as every non-zero real has a unique inverse in $\mathbb{R}^{\times}$ and $\ln : G \to \mathbb{R}^{\times}$ has an inverse function. We will use this in part 2.

   If a and b solve $1 = \ln a \ln b$ then it is easy to see that $b * a$ must also be e.

   Associativity follows from

   $$\begin{aligned}(a * b) * c &= \left(a^{\ln b}\right) * c \\ &= \left(a^{\ln b}\right)^{\ln c} \\ &= a^{\ln b \ln c} \\ &= a^{\ln c \ln b} \\ &= a^{\ln \left(b^{\ln c}\right)} \\ &= a^{\ln (b * c)} \\ &= a * (b * c)\end{aligned}$$

So, G is group. Let us check that it is Abelian. This follows from

$$\begin{aligned}
a * b &= a^{\ln b} \\
&= \left(e^{\ln a}\right)^{\ln b} \\
&= e^{\ln a \ln b} \\
&= e^{\ln b \ln a} \\
&= \left(e^{\ln b}\right)^{\ln a} \\
&= b^{\ln a} \\
&= b * a
\end{aligned}$$

2. Consider $\ln : G \to \mathbb{R}^{\times}$. We know this function has an inverse, the restriction of e^x to $\mathbb{R}^{\times}$. But, is this a homomorphism? Yes, it is!

$$\ln(a * b) = \ln\left(a^{\ln b}\right) = \ln b \ln a$$

2. Let G_1, G_2 be groups.

1. If $H_1 \leq G_1$ and $H_2 \leq G_2$ prove that $H_1 \times H_2 \leq G_1 \times G_2$.
2. TRUE/FALSE : If $H \leq G_1 \times G_2$ then $H = H_1 \times H_2$ for some $H_1 \leq G_1$ and some $H_2 \leq G_2$. Prove your answer!

Solution.

1. Assume $H_1 \leq G_1$ and $H_2 \leq G_2$, thus $H_1 \times H_2$ is non-empty. Let $(h_1, h_2), (g_1, g_2) \in H_1 \times H_2$, then

   $$(h_1, h_2)(g_1, g_2)^{-1} = (h_1, h_2)(g_1^{-1}, g_2^{-1}) = (h_1 g_1^{-1}, h_2 g_2^{-1})$$

   which is an element of $H_1 \times H_2$ because of closure of H_1 and H_2.

2. False: Consider $G = \mathbb{Z} \times \mathbb{Z}$, and H the group generated by $(1, 1)$, which is clearly isomorphic to $\mathbb{Z}$.

   Since a subgroup of G of the form $H_1 \times H_2$ with $H_1, H_2 \leq \mathbb{Z}$ can be isomorphic to $\mathbb{Z}$ only if one of H_1 or H_2 is trivial, then our group H must be contained in $\mathbb{Z}$, which is impossible, as H is generated by $(1, 1)$.

3. If $\phi : S_3 \to \mathbb{Z}_3$ is a group homomorphism, show that $\phi(g) = 0$ for all $g \in S_3$.

Solution. Since S_3 is generated by 2-cycles then we just need to see what ϕ does to these elements. Now, since 2-cycles have order two, then the order of their images under ϕ must have order two or one (divisors of 2). It follows that ϕ must send 2-cycles to the identity of $\mathbb{Z}_3$ because this group does not have element of order 2. Since every generator of S_3 is mapped to 0 then every element in S_3 is mapped to zero under ϕ.

4. Define $f : \mathbb{Z}_{mn} \to \mathbb{Z}_m \times \mathbb{Z}_n$ by $f([x]_{mn}) = ([x]_m, [x]_n)$. Show that f is well-defined, and that f is bijective if and only if $gcd(m, n) = 1$.

Solution. Let $[x]_{mn} = [y]_{mn}$, that is $x = y + mn\alpha$ for some $\alpha \in \mathbb{Z}$. Then,

$$f([x]_{mn}) = ([x]_m, [x]_n) = ([y + mn\alpha]_m, [y + mn\alpha]_n) = ([y]_m, [y]_n) = f([y]_{mn})$$

So, f is well-defined.
In order to see if f is onto we need to check if for $([a]_m, [b]_n) \in \mathbb{Z}_m \times \mathbb{Z}_n$ there is an $[x]_{mn}\mathbb{Z}_{mn}$ such that $f([x]_{mn}) = ([a]_m, [b]_n)$. In other words, we are looking for an $x \in \mathbb{Z}$ that solves the congruences

$$x \equiv a \pmod{m} \qquad\qquad x \equiv b \pmod{n}$$

simultaneously.
We know that such an x exists if $gcd(m, n) = 1$ because of the Chinese Remainder Theorem.
Since $\mathbb{Z}_{mn}$ and $\mathbb{Z}_m \times \mathbb{Z}_n$ have the same number of elements, then assuming $gcd(m, n) = 1$ implies that f is bijective.
If we assume that f is bijective, then we get a unique solution modulo n for the two congruences above. But, if $gcd(m, n) = d \neq 1$ then for x a solution of the congruences above we get another solution (distinct from x modulo mn), namely $y = x + \frac{mn}{d}$. This is a contradiction, so $d = 1$.

5. Let $H \trianglelefteq G$ and for any $g \in G$ define $n_H(g)$ to be the least positive integer such that $g^{n_H(g)} \in H$. Show that $n_H(g)$ divides the order of g.

Solution. Note that we will not use the normality of H for this proof. There are other proofs that might use it, though.
Let $H \leq G$ and for $g \in G$ define $n_H(g)$ as above. Let m be the order of g.
Since $g^m = e$, then $g^m \in H$. It follows that $n_H(g) \leq m$, thus we can use the division (Euclidean) algorithm to get

$$m = n_H(g)q + r$$

where $q, r \in \mathbb{Z}$ and $0 \leq r < n_H(g)$.

Note that

$$g^m = g^{n_H(g)q+r} = g^{n_H(g)q} g^r$$

which implies

$$\left(g^{n_H(g)q}\right)^{-1} g^m = g^r$$

Since both $g^{n_H(g)q}$ and g^m live in H, then so does g^r. If r were non-zero then there would be a positive integer that is less than $n_H(g)$ such that $g^r \in H$. This is a contradiction, so $r = 0$ and thus $n_H(g) \mid m$.

6. Let R be a commutative ring. An element $r \in R$ is called *nilpotent* if $r^n = 0$ for some integer $n > 0$. Prove that $a + b$ is nilpotent if a and b are nilpotent elements of R.

Solution. Assume $a^n = b^m = 0$, then since the ring is commutative we get

$$\begin{aligned}
(a+b)^{m+n} &= \sum_{k=0}^{m+n} \binom{m+n}{k} a^k b^{m+n-k} \\
&= \sum_{k=0}^{n} \binom{m+n}{k} a^k b^{m+n-k} + \sum_{k=n+1}^{m+n} \binom{m+n}{k} a^k b^{m+n-k}
\end{aligned}$$

For $0 \leq k \leq n$, $m + n - k > m$ and thus $b^{m+n-k} = 0$. Thus the first sum is n zero summands. Similarly, for $n + 1 \leq k \leq m + n$ we get $a^k = 0$, which implies that the second sum is also zero. Hence, $(a + b)^{m+n} = 0$.

7. Show that the set of matrices $A \in M_n(\mathbb{R})$ such that $Av = 0$ for some fixed $v \in \mathbb{R}^n$ is a left ideal of $M_n(\mathbb{R})$.

Solution. Fix $v \in \mathbb{R}^n$. Let $I = \{A \in M_n(\mathbb{R});\ Av = 0\}$.
It is clear that the zero matrix is in I, and thus I is non-empty. Now let $A, B \in I$, then

$$(A - B)v = Av - Bv = 0 - 0 = 0$$

which means that $A - B \in I$.
Now let $A \in I$ and $B \in M_n(\mathbb{R})$, then

$$(BA)v = B(Av) = B \cdot 0 = 0$$

which means that $BA \in I$.
It follows that I is a left ideal.

8. Let

$$S = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \mid a, b \in \mathbb{R} \right\} \subset M_2(\mathbb{R})$$

1. Show that S is a subring of $M_2(\mathbb{R})$, the ring of 2×2 matrices with real entries.
2. Show that S and $\mathbb{C}$ are isomorphic rings.

Solution.

1. S is non-empty as the zero matrix and the identity matrix are elements of S. Now take two elements in S and subtract them

   $$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} - \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} = \begin{bmatrix} a-\alpha & b-\beta \\ -(b-\beta) & a-\alpha \end{bmatrix}$$

   which is an element of S.

   Similarly, the product of two elements of S is

   $$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} = \begin{bmatrix} a\alpha - b\beta & a\beta + b\alpha \\ -(b\alpha + a\beta) & a\alpha - b\beta \end{bmatrix}$$

   which is also an element of S.

2. Consider the map $\phi : S \to \mathbb{C}$ defined by

   $$\phi \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = a + bi$$

   Since

   $$\begin{aligned} \phi \begin{bmatrix} a & b \\ -b & a \end{bmatrix} + \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} &= \begin{bmatrix} a+\alpha & b+\beta \\ -(b+\beta) & a+\alpha \end{bmatrix} \\ &= (a+\alpha) + (b+\beta)i \\ &= (a+bi) + (\alpha+\beta i) \end{aligned}$$

   and

   $$\begin{aligned} \phi \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} &= \phi \begin{bmatrix} a\alpha - b\beta & a\beta + b\alpha \\ -(b\alpha + a\beta) & a\alpha - b\beta \end{bmatrix} \\ &= (a\alpha - b\beta) + (a\beta + b\alpha)i \\ &= (a+bi)(\alpha+\beta i) \end{aligned}$$

then ϕ is a homomorphism of rings.

The kernel of ϕ is

$$Ker(\phi) = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} ; \ \phi \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = 0 \right\} = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} ; a + bi = 0 \right\} = \{0\}$$

So, ϕ is one-to-one. Checking onto is easy, as for any $a + bi \in \mathbb{C}$

$$\phi \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = a + bi$$

Part B.

1. Recall that $\mathcal{P}_5$ is the set containing the zero polynomial and all polynomials of degree at most five with real coefficients. Show that the derivative defines a linear transformation from $\mathcal{P}_5$ to itself. Is it onto? Find the matrix for this map in the standard basis.

Solution. Since for any pair of differentiable functions (including polynomials) f and g, and any constant C we have

$$(f+g)' = f' + g' \qquad \textit{and} \qquad (Cf)' = Cf'$$

then the derivative behaves linearly. Moreover, since the derivative of a polynomial of degree at most five is of degree at most four, and $0' = 0$, then the function is from $\mathcal{P}_5$ to $\mathcal{P}_5$. Also, since there is no way to obtain $p(x) = x^5$ as the derivative of a polynomial of degree at most five (its anti-derivative has degree six), then ϕ is not onto.
The standard basis for $\mathcal{P}_5$ is $\mathcal{B} = \{1, x, x^2, x^3, x^4, x^5\}$. Since

$$\phi(1) = 0 \qquad \phi(x) = 1 \qquad \phi(x^2) = 2x \qquad \phi(x^3) = 3x^2 \qquad \phi(x^4) = 4x^3 \qquad \phi(x^5) = 5x^4$$

then the matrix of ϕ with respect to $\mathcal{B}$ is

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2. Show that

$$U = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{R} \text{ and } a + b + c + d = 0 \right\} \subset M_2(\mathbb{R})$$

is a subspace of $M_2(\mathbb{R})$. Find a basis for U.

Solution. Probably the easiest way to do this is to go ahead and re-write U as the span of a set of vectors (which will turn out being its basis).

We use $a+b+c+d=0$ to get $d=-a-b-c$. Thus an element of U looks like

$$\begin{aligned}
\begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{bmatrix} a & b \\ c & -a-b-c \end{bmatrix} \\
&= \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & -b \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c & -c \end{bmatrix} \\
&= a\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}
\end{aligned}$$

Since the coefficients a, b and c can take any values, then U is the span of

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \right\}$$

which makes U a subspace of $M_2(\mathbb{R})$.
The matrices in $\mathcal{B}$ are clearly linearly independent, thus $\mathcal{B}$ is a basis of U.

3. Find a basis for the orthogonal complement of the subspace $W = span\{(1,2,-1,0),(0,1,1,3)\}$ of $\mathbb{R}^4$.

Solution. Any vector that is orthogonal to every element of W must be orthogonal to both $(1,2,-1,0)$ and $(0,1,1,3)$ (and vice-versa). Thus, a vector $(x,y,z,w) \in W^\perp$ must be a solution of the system of equations

$$x+2y-z=0 \qquad\qquad y+z+3w=0$$

Solving for x in the first equation and for w in the second we get

$$x=-2y+z \qquad\qquad w=-\frac{y}{3}-\frac{z}{3}$$

Thus, the vectors $(x,y,z,w) \in W^\perp$ look like

$$\begin{aligned}
(x,y,z,w) &= \left(-2y+z, y, z, -\frac{y}{3}-\frac{z}{3}\right) \\
&= \left(-2y, y, 0, -\frac{y}{3}\right) + \left(z, 0, z, -\frac{z}{3}\right) \\
&= y\left(-2, 1, 0, -\frac{1}{3}\right) + z\left(1, 0, 1, -\frac{1}{3}\right)
\end{aligned}$$

which says, given that y and z are free to roam all over $\mathbb{R}$, that

$$W^\perp = span\left\{ \left(-2,1,0,-\frac{1}{3}\right), \left(1,0,1,-\frac{1}{3}\right) \right\}$$

Since the two vectors that span $W^\perp$ are linearly independent (because one is not a multiple of the other), then they are a basis of $W^\perp$.

4. Let T be the linear transformation of $\mathbb{R}^3$ with standard matrix $\begin{bmatrix} 1 & 5 & 2 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$. Find the matrix of T with respect to the basis $\mathbb{B} = \{(1,1,1), (1,1,0), (1,0,0)\}$.

Solution. Let us denote the standard basis of $\mathbb{R}^3$ by $\mathbb{S}$. The change of basis matrix (or transition matrix) from $\mathbb{B}$ to $\mathbb{S}$ is given by just 'hanging' the vectors of $\mathbb{B}$ to get

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

It follows that the matrix representing T with respect to $\mathbb{B}$ is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 5 & 2 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Since

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

then the matrix representing T with respect to $\mathbb{B}$ is

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 5 & 2 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 2 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & -1 \end{bmatrix}$$

5. Let $F : \mathbb{R}^4 \to \mathbb{R}^3$ be any linear transformation such that

$$\operatorname{Ker} F = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \;\middle|\; \begin{array}{r} x_1 - 2x_2 + x_3 - x_4 = 0 \\ x_1 - x_2 - 2x_3 + x_4 = 0 \\ x_1 - 3x_2 + 4x_3 - 3x_4 = 0 \end{array} \right\}.$$

1. Find the dimension of $\operatorname{Ker} F$ and a basis for it.
2. Give an example of such a linear transformation F.
3. For the example you gave in (b), find a basis for the range of F.

Solution.

1. We need to solve the system given. The matrix that represents this (homogeneous) system is

$$A = \begin{bmatrix} 1 & -2 & 1 & -1 \\ 1 & -1 & -2 & 1 \\ 1 & -3 & 4 & -3 \end{bmatrix}$$

Let us do some row operations in A.

$$\begin{aligned} A &= \begin{bmatrix} 1 & -2 & 1 & -1 \\ 1 & -1 & -2 & 1 \\ 1 & -3 & 4 & -3 \end{bmatrix} && \text{now we subtract } R_1 \text{ from } R_3 \text{ and } R_2 \\ &\to \begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 1 & -3 & 2 \\ 0 & -1 & 3 & -2 \end{bmatrix} && \text{now we add } R_2 \text{ to } R_3\text{, and add } 2R_2 \text{ to } R_1 \\ &\to \begin{bmatrix} 1 & 0 & -5 & 3 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

It follows that $x_1 = 5x_3 - 3x_4$ and that $x_2 = 3x_3 - 2x_4$. Thus the vectors in $Ker(F)$ look like

$$\begin{aligned} (x_1, x_2, x_3, x_4) &= (5x_3 - 3x_4, 3x_3 - 2x_4, x_3, x_4) \\ &= (5x_3, 3x_3, x_3, 0) + (-3x_4, -2x_4, 0, x_4) \\ &= x_3(5, 3, 1, 0) + x_4(-3, -2, 0, 1) \end{aligned}$$

Since x_3 and x_4 have no restrictions then

$$Ker(F) = span\{(5, 3, 1, 0), (-3, -2, 0, 1)\}$$

These vectors are linearly independent, thus they form a basis of $Ker(F)$. $dim(Ker(F)) = 2$.

2. The map $F : \mathbb{R}^4 \to \mathbb{R}^3$ given by

$$F(x_1, x_2, x_3, x_4) = (x_1 - 2x_2 + x_3 - x_4, x_1 - x_2 - 2x_3 + x_4, x_1 - 3x_2 + 4x_3 - 3x_4)$$

3. The range of F can be computed by looking at the column space of the matrix A used in part 1. of this problem. So, now we will do some column operations.

on A.

$$\begin{array}{rcl} A & = & \begin{bmatrix} 1 & -2 & 1 & -1 \\ 1 & -1 & -2 & 1 \\ 1 & -3 & 4 & -3 \end{bmatrix} \qquad \text{now we subtract } C_1 \text{ from } C_3 \text{ and add } C_1 \text{ to } C_4 \\ & \to & \begin{bmatrix} 1 & -2 & 0 & 0 \\ 1 & -1 & -3 & 2 \\ 1 & -3 & 3 & -2 \end{bmatrix} \qquad \text{now we add } \frac{2}{3}C_3 \text{ to } C_4\text{, and } C_3 \text{ to } C_2 \\ & \to & \begin{bmatrix} 1 & -2 & 0 & 0 \\ 1 & -4 & -3 & 0 \\ 1 & 0 & 3 & 0 \end{bmatrix} \end{array}$$

Since $(1, 1, 1) = -\frac{1}{2}(-2, -4, 0) + \frac{1}{3}(0, -3, 3)$, then the column space is

$$span\{(-2, -4, 0), (0, -3, 3)\} = span\{(1, 2, 0), (0, -1, 1)\}$$

These two vectors are linearly independent, thus they form a basis of the range of F.

6. A square matrix B is *skew-symmetric* if $B^T = -B$. Suppose that the square matrix A is skew-symmetric and invertible. Prove that A^{-1} is also skew-symmetric.

Solution. We know that $A^T = -A$ and that A is invertible. Since $(A^T)^{-1} = (A^{-1})^T$, then A^T is invertible. It follows that if we inverse both sides of $A^T = -A$ we get

$$(A^{-1})^T = (A^T)^{-1} = (-A)^{-1} = -(A^{-1})$$

which proves that A^{-1} is also skew-symmetric.

7. Diagonalize the following matrix

$$A = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

Then give a basis of $\mathbb{R}^3$ for which A 'becomes' diagonal.

Solution. We first find the characteristic polynomial of A.

$$\begin{aligned}
\chi_A(\lambda) &= det(A-\lambda I) \\
&= \begin{vmatrix} 1-\lambda & -2 & -1 \\ -1 & 1-\lambda & 1 \\ 1 & 0 & -1-\lambda \end{vmatrix} \\
R_2 \mapsto R_2 + R_3 & \begin{vmatrix} 1-\lambda & -2 & -1 \\ 0 & 1-\lambda & -\lambda \\ 1 & 0 & -1-\lambda \end{vmatrix} \\
C_3 \mapsto C_3 - C_2 & \begin{vmatrix} 1-\lambda & -2 & 1 \\ 0 & 1-\lambda & -1 \\ 1 & 0 & -1-\lambda \end{vmatrix} \\
&= -(1-\lambda)^2(1+\lambda) + 2 - (1-\lambda) \\
&= -(1-\lambda)^2(1+\lambda) + (\lambda+1) \\
&= (1+\lambda)(1-(1-\lambda)^2) \\
&= (1+\lambda)(2\lambda-\lambda^2) = \lambda(1+\lambda)(2-\lambda)
\end{aligned}$$

It is easy to see that $\lambda = 0, -1, 2$ are the eigenvalues of $\chi_A(\lambda)$. Since the eigenvalues are distinct the matrix is diagonalizable.

We know that the base where A 'becomes' diagonal is given by the eigenvectors of A. Let us find them.

For $\lambda = 0$

$$[A|0] = \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ -1 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] \quad \text{now we subtract } R_3 \text{ from } R_1 \text{ and add } R_3 \text{ to } R_2$$

$$\rightarrow \left[\begin{array}{ccc|c} 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right]$$

It follows that the eigenspace of $\lambda = 0$ is $span(1, 0, 1)$.

For $\lambda = -1$

$$[A+I|0] = \left[\begin{array}{ccc|c} 2 & -2 & -1 & 0 \\ -1 & 2 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right] \quad \text{now we subtract } 2R_3 \text{ from } R_1 \text{ and add } R_3 \text{ to } R_2$$

$$\rightarrow \left[\begin{array}{ccc|c} 0 & -2 & -1 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right]$$

It follows that the eigenspace of $\lambda = -1$ is $span(0, 1, -2)$.
For $\lambda = 2$

$$[A - 2I|0] \quad = \quad \left[\begin{array}{ccc|c} -1 & -2 & -1 & 0 \\ -1 & -1 & 1 & 0 \\ 1 & 0 & -3 & 0 \end{array}\right] \qquad \text{now we add } R_3 \text{ to } R_1 \text{ and } R_2$$

$$\rightarrow \left[\begin{array}{ccc|c} 0 & -2 & -4 & 0 \\ 0 & -1 & -2 & 0 \\ 1 & 0 & -3 & 0 \end{array}\right]$$

It follows that the eigenspace of $\lambda = 2$ is $span(3, -2, 1)$.
Thus, the basis of $\mathbb{R}^3$ where A becomes

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

is

$$\mathcal{B} = \{(1, 0, 1), (0, 1, -2), (3, -2, 1)\}$$

8. Consider the subspace $U = \{(x, y, z) \in \mathbb{R}^3 |\, 2x - y - 3z = 0\} \subset \mathbb{R}^3$ and the set of vectors $S = \{(1, -1, 1), (4, 2, 2)\} \subset \mathbb{R}^3$.

1. Complete S to a basis in $\mathbb{R}^3$.
2. Show that $U = span(S)$.

Solution.

1. If we show that $U = span(S)$ (part 2. of this problem), then a basis for the the orthogonal complement of U would complete S to a basis of $\mathbb{R}^3$. Since U is a plane through the origin in $\mathbb{R}^3$, then its orthogonal complement is given by its normal vector, which can be obtained by just looking at the coefficients of the equation that defines the plane, namely $N = (2, -1, -3)$.

   So, $S \cup \{(2, -1, -3)\}$ is a basis of $\mathbb{R}^3$.

2. We plug the vectors in S into $2x - y - 3z = 0$ (the equation that defines U) to check that these vectors belong to U, we get

   $$2(1) - (-1) - 3(1) = 0 \qquad\qquad 2(4) - (2) - 3(2) = 0$$

   Thus $S \subset U$. Since the two vectors in S are linearly independent and U is a plane (dimension 2), then the vectors in S must span all U.