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**Part A.** Solve **five** of the following eight problems:

1. True or False (prove or give a counterexample):
   Let $A, N, G$ be groups such that $A \trianglelefteq N$ and $N \trianglelefteq G$, then $A \trianglelefteq G$.

2. Suppose you are given the operation $*$ on the set $G = \{x \in \mathbb{R} \mid x \neq -1\}$, defined by $a * b = ab + a + b$. Show that under this operation $G$ is a group.

3. Suppose that $\phi : \mathbb{Z}_{50} \to \mathbb{Z}_{15}$ is a group homomorphism with $\phi(7) = 6$.
   (a) Determine $\phi(x)$.
   (b) Determine the image of $\phi$.
   (c) Determine the kernel of $\phi$.

4. True or False (prove or give a counterexample):
   If a ring $R$ has more than two idempotent elements ($e$ is idempotent if $e^2 = e$), then $R$ is not an integral domain.

5. Let $N = \langle([2], (123))\rangle \lhd \mathbb{Z}_4 \times S_3$.
   (a) Find the order of the factor group $(\mathbb{Z}_4 \times S_3)/N$.
   (b) Find the order of the element $([3], (12))N$ in $(\mathbb{Z}_4 \times S_3)/N$. Is $(\mathbb{Z}_4 \times S_3)/N$ cyclic?

6. Let $G$ be a finite group of odd order. Prove that every element in $G$ has a square root (so you have to show that for all $g \in G$, there exists $x \in G$ such that $x^2 = g$).

   Hint: Show that the map $\theta : G \to G : g \to g^2$ is one-to-one.

7. Suppose that $a$ and $b$ are group elements that commute and have orders $m$ and $n$. If $\langle a\rangle \cap \langle b\rangle = \{e\}$, prove that the group contains an element whose order is the least common multiple of $m$ and $n$. Show that this need not be true if $a$ and $b$ do not commute.

8. Let $R$ be a ring with 1, and let $U(R)$ be the set of all units in $R$.
   (a) Show that $U(R)$ is a group.
   (b) If $I$ is an ideal (left, right, or two-sided) in $R$ such that $I \cap U(R) \neq \emptyset$, show that $I = R$.

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**Part B is on the back!!!**

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**Part B.** Solve **five** of the following eight problems :

1. Consider a system of equations $A\mathbf{x} = \mathbf{b}$ where $A$ is $n \times k$. Give a proof or counterexample for the following:

   (a) For given $A$ and $\mathbf{b}$, if $n = k$ then there is always at most one solution.

   (b) For given $A$ and $\mathbf{b}$, if $n > k$ then there is always at least one solution.

   (c) For given $A$, if $n < k$ then there exists a vector $\mathbf{b}$ for which the system has no solution.

2. Let $V$ be an $n$-dimensional vector space and $T : V \to V$ a linear transformation such that the image and kernel of $T$ are identical.

   (a) Prove that $n$ is even.

   (b) Give an example of such a linear transformation $T$.

3. Let $P_2$ be the set of all real polynomials of degree at most 2. It is given that the map

$$f : P_2 \to P_2 : p(x) \to p(x) - p'(x) + p''(x)$$

   is a linear transformation and that $\beta = \{1, 1+x, 1+x+x^2\}$ is a basis for $P_2$. Find the matrix representation of $f$ with respect to the basis $\beta$.

4. Let $A$ be a square matrix with the property that the sum of the elements in each of its columns is 1. Show that $\lambda = 1$ is an eigenvalue of $A$.

5. Let $\theta : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that $\theta(2,-1) = (1,0,1)$ and $\theta(-5,3) = (0,-1,1)$. Find an expression for $\theta(x,y)$.

6. Let $W_1$ and $W_2$ be subspaces of a finite-dimensional vector space $V$. Prove the following:

   (a) $W_1 + W_2 = \{w_1 + w_2 : w_1 \in W_1, w_2 \in W_2\}$ is a subspace of $V$.

   (b) $W_1 \cap W_2$ is a subspace of $V$.

   (c) $dim\,(W_1) + dim\,(W_2) = dim\,(W_1 + W_2) + dim\,(W_1 \cap W_2)$.

7. Find all values of $\alpha$ for which the following matrix is nonsingular

$$\begin{pmatrix} -2 & \alpha & 3 \\ 1 & 2 & \alpha \\ 1 & 11 & 18 \end{pmatrix}.$$

8. Suppose that $\mathbf{v}_1$ and $\mathbf{v}_2$ are arbitrary vectors in $\mathbb{R}^n$. Prove that

$$\text{span}\,\{\mathbf{v}_1, \mathbf{v}_2\} = \text{span}\,\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2\}.$$