Part A.

1. False: Consider $G = A_4$, N to be the Klein group

$$N = \{e, (12)(34), (13)(24), (14)(23)\}$$

and $A =< (12)(34) >$. It is easy to see that $A \trianglelefteq N \trianglelefteq G$, but A is not normal in G because $(123)A(123)^{-1} =< (14)(23) > \neq A$.

2. We need to check all the axioms of a group. First we realize that the product of any two elements of $\mathbb{R}$ yields an element of $\mathbb{R}$, thus closure is (almost) granted. We just need to check that the product of any two elements in G is not -1. If $a * b = -1$ then

$$-1 = a + b(1 + a) \qquad \textit{or} \qquad -(1 + a) = b(1 + a)$$

which forces either $1 + a = 0$ (impossible as $a \neq -1$) or $b = -1$ (then again impossible). It follows that $a * b \neq -1$, and thus we have closure for G.

In the search for an identity e we set $a * e = a$, we get $a = a + e + ae$, which means that $e(1 + a) = 0$. But, since $a \neq -1$ then $e = 0$. So, $e = 0$ is the only candidate to be the identity for this product... but it is easy to check that $a * 0 = 0 * a = a$ for all $a \in G$, so we have found the identity.

Now let us find the inverse of an element a, we want to find b such that $a*b = 0$. It is ease to see that $a + b + ab = 0$ forces

$$b = -\frac{a}{1 + a}$$

which then again needs $a \neq -1$ to be well defined. Now note that this fraction can never be -1 (as a and $1+a$ are never equal to each other), thus the (right) inverse of any element in G is well defined and an element of G. Now it is easy to check that $a*b = b*a = 0$m and thus every element in G has a double-sided inverse.

The only thing left to check is associativity. This follows simply from

$$\begin{aligned} a*(b*c) &= a*(b+c+bc) \\ &= a+(b+c+bc)+a(b+c+bc) \\ &= a+b+c+bc+ab+ac+abc \\ &= (a+b+ab)+c+(a+b+ab)c \\ &= (a+b+ab)*c \\ &= (a*b)*c \end{aligned}$$

Hence, G is a group. Moreover, it is easy to see that it is an Abelian group.

3. First note that since $gcd(50, 7) = 1$ then 7 is a generator of $(\mathbb{Z}_{50}, +)$. Thus defining $\phi(7)$ will determine uniquely ALL the values ϕ takes in $\mathbb{Z}_{50}$.

   (a) In order to find the images we need to write any element in $\mathbb{Z}_{50}$ as a multiple (modulo 50) of 7. Since,

   $$50 + 7 \cdot (-7) = 1$$

   then, using the homomorphism properties of ϕ we get

   $$\begin{aligned} \phi(1) &= \phi(50 + 7 \cdot (-7)) \\ &= \phi(50) + \phi(7 \cdot (-7)) \\ &= \phi(0) - 7\phi(7) \\ &= -7\phi(7) \\ &= -42 \\ &= 3 \end{aligned}$$

   where the last step was reduction modulo 15 (the images of ϕ live in $\mathbb{Z}_{15}$). It follows that for every integer x,

   $$\phi(x) = x\phi(1) = 3x \pmod{15}$$

   (b) Since 3 divides 15 then the multiples of 3 (modulo 15) yields a proper subgroup of $\mathbb{Z}_{15}$, which is

   $$\phi(\mathbb{Z}_{50}) = \{0, 3, 6, 9, 12\}$$

   which is isomorphic to $\mathbb{Z}_5$.

(c) Since $15 = 3 \cdot 5$, then any x that has a five in its prime factorization (divisible by 5) will be in $Ker(\phi)$. Hence, the kernel is the set of multiples of 5 modulo 50, that is

$$Ker(\phi) = \{0, 5, 10, 15, 20, 25, 30, 35, 40, 45\}$$

which is isomorphic to $\mathbb{Z}_{10}$.

Note that the first isomorphism theorem would say, in this case, something like

$$\mathbb{Z}_{50}/\mathbb{Z}_{10} \cong \mathbb{Z}_5$$

4. TRUE: $e^2 = e$ implies $e(e-1) = 0$, the two obvious solutions are $e = 0, 1$, if there were another solution, then e and $e-1$ would be both non-zero with product equal to zero. Hence, e and $e-1$ would be zero divisors.

5. Since $[2]$ has order 2 in $\mathbb{Z}_4$ and (123) has order 3 in S_3, then $\sigma = ([2], (123))$ has order 6, and thus $|N| = 6$ (and cyclic).

   (a) First of all notice that N is normal in $G = \mathbb{Z}_4 \times S_3$, and that $|G| = 4 \cdot 6 = 24$. It follows that G/N has $24/6 = 4$ elements.

   (b) First notice that $([3], (12)) \notin N$. Since $2 \cdot [3] = [2]$ and $(12)^2 = e$, then $([3], (12))^2 = ([2], e) = ([2], (123))^3$, which lives in N. It follows that the order of $([3], (12))N$ in G/N is 2.
   Now, note that $([3], (123)) \notin N$, and that

   $$([3], (123))([3], (12))^{-1} = ([0], (13)) \notin N$$

   and thus $([3], (123))N \neq ([3], (12))N$. Moreover, since

   $$([3], (123))^2 = ([2], (132)) = ([2], (123))^5 \in N$$

   then G/N contains at least two elements of order 2, thus it cannot be cyclic.

6. Consider $\theta : G \to G$ defined by $\theta(g) = g^2$. Let $|G| = n < \infty$. We want to show that θ is onto.

   Using that $gcd(n, 2) = 1$ we get $\alpha, \beta \in \mathbb{Z}$ such that $1 = 2\alpha + n\beta$, then

   $$x = x^{2\alpha+n\beta} = x^{2\alpha}x^{n\beta} = x^{2\alpha}(x^n)^\beta = x^{2\alpha} = (x^\alpha)^2$$

   for all $x \in G$. So, $\theta(x^\alpha) = x$.

7. Let G be the group that contains the elements a and b.

   Consider $x = (ab)^{gcd(m,n)} \in G$. Since $lcm(m,n)gcd(m,n) = mn$ then

$$x^{lcm(m,n)} = \left((ab)^{gcd(m,n)}\right)^{lcm(m,n)} = ab^{mn} = a^m b^n = e$$

   Hence, the order of x is a divisor of $lcm(m,n)$. If the order d of x were less than that, then

$$e = x^d = \left((ab)^{gcd(m,n)}\right)^d = a^{dgcd(m,n)} b^{dgcd(m,n)}$$

   which forces that the element $a^{dgcd(m,n)} = b^{-dgcd(m,n)}$ is in the intersection of the groups generated by a and b. This forces $a^{dgcd(m,n)} = b^{-dgcd(m,n)} = e$, but this implies that $n|d\ gcd(m,n)$ and $m|d\ gcd(m,n)$, but since $d < lcm(m,n)$ we get a contradiction. Hence, the order of x is exactly $lcm(m,n)$.

   For the counterexample, consider $a = (123)$ and $b = (12)$, both elements in S_3. Since S_3 is not cyclic then there is no element of order 6 in this group.

8. Let R be a ring with 1, and $U(R)$ be the set of all its units.

   (a) We want to show that $U(R)$ is a multiplicative group. Since R has a 1 and associativity of the multiplication is inherited from R, then we just need to check closure under multiplication and inverses.

   Let $r, s \in U(R)$, then $ar = ra = 1$ and $bs = sb = 1$ for some $a, b \in R$. Then,

$$(ba)(rs) = b(ar)s = b \cdot 1 \cdot s = 1 \qquad (rs)(ba) = r(sb)a = r \cdot 1 \cdot a = 1$$

   which means that rs is invertible in R.

   Closure for inverses is straight out of the definition of inverses.

   (b) Let $r \in I \cap U(R)$, then there is an element $a \in R$ such that $ar = ra = 1$. But since I is an ideal (left/right/double-sided), then this forces that $1 \in I$, which immediately implies that all the elements in the ring are contained in I (because of the "absorption" property of ideals).

Part B.

1. For these solutions consider a system of equations $A\mathbf{x} = \mathbf{b}$ where A is $n \times k$.

   (a) False. For $n = k = 2$, the system

   $$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

   has infinitely many solutions (the subspace spanned by $(1, 1)$).

   (b) False. For $n = 3$ and $k = 2$, the system

   $$\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

   has no solutions (the first two equations force $x = y = 0$ and the third asks for $x + y = 1$).

   (c) True. Note that in this case no $\mathbf{b}$ is given, thus we need to look at this in a different way. Since A has more columns than rows, then the row dimension will always be less than the number of columns. But the row dimension is equal to the column dimension (and this is less than k), which forces the columns of A to be linearly dependent, and thus not a basis of $\mathbb{R}^k$.

   Now, the search for an $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$ can be read as the search for coefficients (the components of $\mathbf{x}$) such that $\mathbf{b}$ is a linear combination of the columns of A. But since the column dimension is less than k then there is always at least one $\mathbf{b}$ that will not be written as a linear combination of the columns of A. This means that for that $\mathbf{b}$ the system has no solutions.

2. Let V be an n-dimensional vector space and $T : V \to V$ a linear transformation such that the image and kernel of T are identical.

   (a) Since V is finitely dimensional, then we can use the first isomorphism theorem to get a "dimension formula", that is

   $$dim(V) = dim(Ker(T)) + dim(Im(T))$$

   Since the kernel and image are identical, then they have the same dimension k, and thus $n = 2k$.

(b) Consider $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (0, x)$. This map has $Ker(T) =< (0, 1) >= Im(T)$.

3. We compute the images of the elements of β,

$$f(1) = 1 - 0 + 0 = 1 \qquad f(1 + x) = (1 + x) - 1 + 0 = x$$

$$f(1 + x + x^2) = (1 + x + x^2) - (1 + 2x) + 2 = 2 - x + x^2$$

It follows that the matrix that represents f from the basis β to the standard basis of P_2 is given by

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

The change of basis matrix from basis β to the standard basis of P_2 is

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, the matrix that represents f from basis β to basis β is

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

4. Let A be an $n \times n$ matrix such that the sum of the elements in each of its columns is 1. Note that the transpose of A is such such that the sum of the elements in each of its rows is 1. It follows that if we consider the vector v with 1's in all its components. The product $A^T v$ is equal to v because the components of the product are the sums of the elements in each row of A^T. Thus v is an eigenvector of A^T with eigenvalue 1. But we know that an eigenvalue of A^T is also an eigenvalue of A, and so we are done proving that 1 is an eigenvalue of A.

   In fact if λ is an eigenvalue of A^T then

$$det(A^T - \lambda Id) = 0$$

but

$$(A^T - \lambda Id)^T = (A^T)^T - (\lambda Id)^T = A - \lambda Id$$

and thus

$$det(A - \lambda Id) = det(A^T - \lambda Id) = 0$$

which means that λ is also an eigenvalue of A.

5. Consider $\theta : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that $\theta(2, -1) = (1, 0, 1)$ and $\theta(-5, 3) = (0, -1, 1)$.

   Let us write (x, y) as a linear combination of $(2, -1)$ and $(-5, 3)$.

   $$(x, y) = \alpha(2, -1) + \beta(-5, 3) = (2\alpha - 5\beta, -\alpha + 3\beta)$$

   which yields

   $$x = 2\alpha - 5\beta \qquad\qquad y = -\alpha + 3\beta$$

   and thus $\beta = x + 2y$ and $\alpha = 3x + 5y$. It follows that

   $$(x, y) = (3x + 5y)(2, -1) + (x + 2y)(-5, 3)$$

   and thus

   $$\begin{aligned} \theta(x, y) &= (3x + 5y)\theta(2, -1) + (x + 2y)\theta(-5, 3) \\ &= (3x + 5y)(1, 0, 1) + (x + 2y)(0, -1, 1) \\ &= (3x + 5y, -x - 2y, 4x + 7y) \end{aligned}$$

6. Let W_1 and W_2 be subspaces of a finite-dimensional vector space V.

   (a) We first check that zero is in $W_1 + W_2$, this is clear because $0 + 0 = 0$ and $0 \in W_1$, $0 \in W_2$.

   Now we check closure for sum. Let $w_1 + w_2$ and $v_1 + v_2$ be two elements of $W_1 + W_2$.Then

   $$(w_1 + w_2) + (v_1 + v_2) = (w_1 + v_1) + (w_2 + v_2)$$

   which is in $W_1 + W_2$ because $w_1 + v_1 \in W_1$ and $w_2 + v_2 \in W_2$ (closure of addition in these subspaces).

   Finally we check closure under scalar multiplication. Let $w_1 + w_2$ be an element of $W_1 + W_2$, and $\alpha \in \mathbb{R}$. Then,

   $$\alpha(w_1 + w_2) = \alpha w_1 + \alpha w_2$$

   which is in $W_1 + W_2$ because $\alpha w_1 \in W_1$ and $\alpha w_2 \in W_2$ (closure of scalar multiplication in these subspaces).

(b) We first check that zero is in $W_1 \cap W_2$, this is clear because $0 \in W_1$ and $0 \in W_2$.

Now we check closure for sum. Let w and v be two elements of $W_1 \cap W_2$, then $w+v \in W_1$ and $w+v \in W_2$ (closure of addition in these subspaces), and thus $w+v \in W_1 \cap W_2$.

Finally we check closure under scalar multiplication. Let w be an element of $W_1 \cap W_2$, and $\alpha \in \mathbb{R}$, then $\alpha w \in W_1$ and $\alpha w \in W_2$ (closure of scalar multiplication in these subspaces), and thus $\alpha w \in W_1 \cap W_2$.

(c) We first take a basis of $W_1 \cap W_2$, then we complete this basis to two other bases, one for W_1 and one for W_2. This can always be done by a known theorem (which requires the axiom of choice to be proved).

So, let

$$\{w_1, w_2, \cdots, w_k, w_{k+1}, \cdots, w_n\}$$

be a basis of W_1 (dimension of W_1 is n),

$$\{w_1, w_2, \cdots, w_k, v_{k+1}, v_{k+2}, \cdots, v_m\}$$

be a basis for W_2 (dimension of W_2 is m), and

$$\{w_1, w_2, \cdots, w_k\}$$

be a basis for $W_1 \cap W_2$ (dimension of $W_1 \cap W_2$ is k),

Since the union of the bases of W_1 and W_2 is a spanning set of $W_1 + W_2$, then

$$span\{w_1, w_2, \cdots, w_k, w_{k+1}, \cdots, w_n, v_{k+1}, v_{k+2}, \cdots, v_m\} = W_1 + W_2$$

But this set (with $n + (m - k)$ elements) must be linearly independent otherwise we get a contradiction with either the given bases being linearly dependent or we would find vectors not on the basis of $W_1 \cap W_2$ that should be in the intersection. So, the dimension of $W_1 + W_2$ is $n + (m - k)$, done.

7. We need to compute the determinant of this matrix

$$
\begin{aligned}
\begin{vmatrix} -2 & \alpha & 3 \\ 1 & 2 & \alpha \\ 1 & 11 & 18 \end{vmatrix} &= \begin{vmatrix} -2 & \alpha & 3 \\ 1 & 2 & \alpha \\ 0 & 9 & 18-\alpha \end{vmatrix} && \text{by subtracting } R_2 \text{ from } R_3 \\
&= \begin{vmatrix} 0 & \alpha+4 & 3+2\alpha \\ 1 & 2 & \alpha \\ 0 & 9 & 18-\alpha \end{vmatrix} && \text{by adding } 2R_2 \text{ to } R_1 \\
&= -\begin{vmatrix} \alpha+4 & 3+2\alpha \\ 9 & 18-\alpha \end{vmatrix} && \text{expansion into cofactors, using the first column} \\
&= -\left[(\alpha+4)(18-\alpha)-(3+2\alpha)(9)\right] \\
&= \alpha^2+4\alpha-45 \\
&= (\alpha-5)(\alpha+9)
\end{aligned}
$$

So, the values for which the matrix is nonsingular are $\alpha \neq 5, -9$.

8. Since $\mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{v}_1 - \mathbf{v}_2$ are linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$, then the subspace generated by the first two vectors is a subset of the subspace spanned by the latter two.

   Now we will show that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linear combinations of $\mathbf{v}_1+\mathbf{v}_2$ and $\mathbf{v}_1-\mathbf{v}_2$, and thus we will get the other inclusion that is needed to get

$$\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\} = \operatorname{span}\{\mathbf{v}_1+\mathbf{v}_2, \mathbf{v}_1-\mathbf{v}_2\}.$$

   It is easy to check that

$$\mathbf{v}_1 = \frac{1}{2}(\mathbf{v}_1+\mathbf{v}_2) + \frac{1}{2}(\mathbf{v}_1-\mathbf{v}_2)$$

$$\mathbf{v}_2 = \frac{1}{2}(\mathbf{v}_1+\mathbf{v}_2) - \frac{1}{2}(\mathbf{v}_1-\mathbf{v}_2)$$

   We are done.