
Part A. Solve **five** of the following eight problems:

1. Let $G = \{x \in \mathbb{R} \mid x \neq -1\}$. For $x,\, y \in G$ let $x * y = x + y + xy$. Prove that $*$ is a binary operation on G and that $(G, *)$ is a group.

2. Consider the group S_n of permutations on n elements ($n \geq 3$), and let A_n denote the set of even permutations in S_n. Prove that A_n is a normal subgroup of S_n.

3. Prove that there is no homomorphism from $\mathbb{Z}_4 \oplus \mathbb{Z}_{12}$ onto $\mathbb{Z}_8 \oplus \mathbb{Z}_6$.

4. (a) Let G be a group, and let $a \in G$. Prove that the function $\phi_a : G \to G$, defined by $\phi_a(x) = axa^{-1}$ for all $x \in G$ is an automorphism of G (it is called the *inner automorphism of* G *induced by* a).
   (b) Let G be a group and $Inn(G) = \{\phi_a \,|\, a \in G\}$ be the set of all inner automorphisms of G. Prove that $Inn(G)$ is a group under the operation of function composition.

5. (a) Let R be a commutative ring with unity 1 and let I be an ideal of R. Prove that $r + I$ is a unit (invertible) in R/I if and only if there is an element s in R such that $rs - 1 \in I$.
   (b) Show that the ring $\mathbb{Z} \oplus \mathbb{Z}$ has infinitely many zero-divisors.
   (c) Find all units in the ring $\mathbb{Z} \oplus \mathbb{Z}$.

6. Let R be a ring. Recall that $a \in R$ is called a *nilpotent* element if $a^n = 0$ for some $n \in \mathbb{Z}^+$ and it is called an *idempotent* element if $a^2 = a$.
   (a) Show that if a is an idempotent, then $1 - a$ is also an idempotent.
   (b) If $f : R \to S$ is a ring homomorphism and $a \in R$ is nilpotent, prove that f carries the element a to a nilpotent element in the ring S.
   (c) If R is an integral domain, prove that the only idempotents in R are 0 and 1.
   (d) Show that $a \in R$ is a zero-divisor if and only if $aba = 0$ for some $b \neq 0$.

7. Let G be a group and let $Z(G) = \{g \in G \,|\, gx = xg,\ \forall x \in G\}$.
   (a) Show that if $G/Z(G)$ is cyclic then G is Abelian.
   (b) Show that if G is non-Abelian with $|G| = p^3$, where p is a prime, and $Z(G) \neq \{e\}$, then $|Z(G)| = p$.

8. Suppose G is a finite cyclic group of order at least 3. Prove that G has an even number of generators.

Part B is on the back!!!

Part B. Solve **five** of the following eight problems :

1. Let W be the subspace of $\mathbb{R}^4$ with basis

$$\left\{ \begin{pmatrix} 1 \\ 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix} \right\}.$$

   Find an orthonormal basis for W.

2. Suppose that A is a square matrix of size n and $B = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, ..., \mathbf{x}_n\}$ is a basis of $\mathbb{R}^n$. Show that if A is nonsingular, then $C = \{A\mathbf{x}_1, A\mathbf{x}_2, A\mathbf{x}_3, ..., A\mathbf{x}_n\}$ is a basis of $\mathbb{R}^n$.

3. Find the kernel and range of the linear transformation represented by

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

4. Given the matrix

$$A = \begin{pmatrix} 3 & -5 \\ 1 & -3 \end{pmatrix},$$

   find A^{20}.

5. Let V be an n-dimensional vector space and W a subspace of V . Let $B' = \{\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_m\}$ be a basis for W. Prove that there exist vectors $\mathbf{b}_{m+1}, ..., \mathbf{b}_n$ in V such that $B = \{\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_m, \mathbf{b}_{m+1}, ..., \mathbf{b}_n\}$ is a basis for V.

6. Find all values of λ so that the following matrix is nonsingular:

$$\begin{pmatrix} -2 & \lambda & 3 \\ 1 & 2 & \lambda \\ 1 & 11 & 18 \end{pmatrix}.$$

7. Given the 2×3 matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix},$$

   (a) For which vectors $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ does $A\mathbf{x} = \mathbf{b}$ have a solution?

   (b) Solve $A\mathbf{x} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$.

   (c) What is the rank of A?

8. Consider P_2 with the standard basis $B = \{1, x, x^2\}$. Determine $p(x) = 3 - x + 2x^2$ with respect to the basis $B' = \{1, x + 1, x^2 + x + 1\}$.