
Part A. Solve **five** of the following eight problems:

1. Prove that the set
$$M = \left\{ \begin{bmatrix} m & n \\ 2n & m \end{bmatrix} ;\ m, n \in \mathbb{Z} \right\}$$
is isomorphic, as a ring, to the ring $\mathbb{Z}[2] = \{m + n\sqrt{2} \,|\, m, n \in \mathbb{Z}\}$.

2. **(a)** How many elements of order 7 does the group $\mathbb{Z}_{21} \oplus \mathbb{Z}_{35}$ have? Justify your answer.
(b) How many cyclic subgroups of order 7 does the group $\mathbb{Z}_{21} \oplus \mathbb{Z}_{35}$ have? Justify your answer.

   Note: You may prefer to use the notation $\mathbb{Z}_{21} \times \mathbb{Z}_{35}$ instead of $\mathbb{Z}_{21} \oplus \mathbb{Z}_{35}$. You are free to do so.

3. **(a)** Find all group homomorphisms from $\mathbb{Z}_{42}$ to $\mathbb{Z}_{12}$ (give formulas for each). How many homomorphisms are there from $\mathbb{Z}_{42}$ *onto* $\mathbb{Z}_{12}$?
(b) Are all the group homomorphisms in part (a) also ring homomorphisms?

4. Let $G = GL(2, \mathbb{R})$ and $H = \{A \in G ;\ \det(A) = 3^k,\ k \in \mathbb{Z}\}$. Prove that H is a subgroup of G. Moreover, show that $H \trianglelefteq G$.

5. Let $n \in \mathbb{N}$. Prove that all ideals in $\mathbb{Z}_n$ are principal.

6. Let R be a commutative ring with one. Prove that R is a field if and only if $\{0\}$ and R are the only (two-sided) ideals in R.

7. **(a)** Suppose that H is a normal subgroup of G with $[G : H] = 24$ and $|H| = 11$. If $x \in G$ and $x^{11} = e$, prove that $x \in H$.
(b) Let $H = \{(1), (12)(34)\}$. Prove or disprove: H is normal in A_4.

8. Let G be a group. Assume that G is isomorphic to all its non-trivial subgroups. Prove that G is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}_p$, for some prime p.

Part B is on the back!!!

Part B. Solve **five** of the following eight problems :

*Notation: **(a)** $M_n(K)$ is the set of $n \times n$ matrices with entries in K.*
***(b)** $\mathcal{P}_n$ is the set of all polynomials (with real coefficients) with degree at most n (including the zero polynomial).*

1. Let $\mathbf{v}$ be a fixed vector in $\mathbb{R}^n$, where $n > 1$. Assume that $M \in M_n(\mathbb{R}) \setminus \{0\}$ is such that $M\mathbf{v} = \mathbf{0}$. Prove that there exists a matrix $N \in M_n(\mathbb{R})$ such that $(MN)\mathbf{v} \neq \mathbf{0}$.

2. Define $T : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ by $T(A) = A + A^T$.
   (a) Show that T is a linear transformation.
   (b) Show that the range of T is the set of all B in $M_2(\mathbb{C})$ with the property that $B^T = B$.
   (c) Describe the kernel of T.

3. Recall that the *trace* of a square matrix is the sum of the entries on the main diagonal. Let W be the subset of $M_2(\mathbb{R})$ of matrices with trace equal to 0.
   (a) Show that W is a subspace of $M_2(\mathbb{R})$.
   (b) Let $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\} \subseteq W$. Show that S is a basis for W.

4. Let V be the space $\mathcal{C}[0,1]$ of real-valued continuous functions defined on $[0,1]$, and consider the inner product on V defined by
$$\langle f, g \rangle = \int_0^1 f(t)g(t)\,dt.$$
   Let W be the subspace spanned by the polynomials $p_1(t) = 1, p_2(t) = 2t-1$, and $p_3(t) = 12t^2$. Use the Gram-Schmidt process to find an orthogonal basis for W.

5. Define $T : \mathcal{P}_2 \to \mathbb{R}^2$ by $T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(2) \end{bmatrix}$.
   (a) Show that T is a linear transformation.
   (b) Find the *kernel* of T, <u>and</u> a basis for it.
   (c) Find the matrix for T relative to the standard bases for $\mathcal{P}_2$ and $\mathbb{R}^2$ respectively.

6. Let A be a square matrix. Prove or disprove the following statements:
   (a) The matrices A and A^T have the same eigenvalues, counting multiplicities.
   (b) If A is an $n \times n$ diagonalizable matrix, then each vector in $\mathbb{R}^n$ can be written as a linear combination of eigenvectors of A.
   (c) If A is diagonalizable, then the columns of A are linearly independent.

7. Let $T : \mathcal{P}_1 \to \mathcal{P}_1$ be a linear transformation such that $T(a + bt) = (-2a + b) + (a + 2b)t$. Find the eigenvalues of T. Then choose one of the eigenvalues and find a basis for the corresponding eigenspace.

8. Show that $B = \{t - 1, t + 1\}$ forms a basis for $\mathcal{P}_1$, <u>and</u> find the change-of-coordinates (i.e. change of basis) matrix from the standard basis $C = \{1, t\}$ to the basis B.