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CSET II

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Contents

Preface

This set of lecture notes cover most of the topics you need to study when you prepare to take the Geometry and Statistics CSET (A.K.A. CSET II). Most subjects are looked at in a very deep but straightforward way, which means that these notes might seem dry and too abstract. The idea is that once you understand the concepts covered in these notes then you can spend a good amount of time doing busy work solving examples and practice tests so you can succeed in your test. Moreover, if you understand concepts instead of just knowing how to solve problems then you will be able to teach Geometry at a high level, and be a teacher who can inspire students by doing things the right way.

It is my believe that just reading these notes is not enough to pass the test, as quite a few topics need to be discussed and studied with the help of an instructor, or somebody else who knows, and understands, the material completely.

Good luck preparing for your test,

O.V.

Chapter 1
Proofs and Constructions

One of the important parts of the CSET's is the constructed response part. These questions weight four times a regular question, and approximate 40% of the whole test. Since, most people agree, one should score above 70% to pass the test then failing the constructed response questions means sure test failing. Moreover, besides the four constructed response questions there are other questions that measure your ability of explaining the reason something is true, and not necessarily to know 'how to solve' something, and this is exactly what knowing how to prove something will allow you to do well.

The need for knowing how to write procedures, and express ideas, in proper mathematics makes this first chapter a very important one. In it you will be introduced to proofs, which is the formal name of what colloquially one could call 'a thorough and complete explanation, or deduction, of a fact by using logic'.

Probably the best way to get familiarized with proofs is to read a lot of them and do a lot of them. So, when you read this book do not think proofs as something to skip, but as examples. Doing this will take the fear you might have of proofs out of you, and it will help you to understand better the concepts you need to know.

As mentioned above, the more you read and do proofs, the better. For the reading part, this book supplies many proofs, at different levels of difficulty, and about many different subjects. Most of the times whenever an important result is mentioned, it is accompanied with its proof. Please read these proofs, understand them, enjoy them.

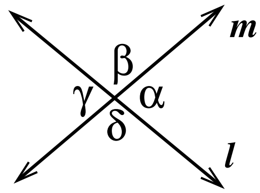
For the doing part, there are many exercises at the end of chapters. Most of them are of the form "prove that...". Please practice your proving skills as, even if you get lucky and not get many proofs in your test, the understanding of where things come from will help you to apply those results in other, more computational, problems.

We will look at two proving techniques, a third one will be discussed in chapter 10.

1.1 Direct proof

In a perfect world, all proofs will be obtained as a logic succession of deductions that will lead the argument from the information given (called hypothesis) directly to the desired result (called theorem, lemma, proposition, goal, etc). Since we do not live in a perfect world then we will need to learn later about contradiction, but for now let us see how these chains of deductions (AKA direct proofs) can be constructed.

Theorem 1.1 (Vertical Angle Theorem). *In the following picture, in which four angles are formed by two intersecting lines*



it is always true that $\alpha = \gamma$ *and that* $\beta = \delta$.

Proof. Note that $\alpha+\beta = 180°$ and that $\gamma+\beta = 180°$. Since both of these sums equal 180° then we can set them equal to each other. We get

$$\alpha+\beta = \gamma+\beta$$

We subtract β both sides and get $\alpha = \gamma$, which is what we wanted.
In a similar way we obtain $\beta = \delta$. □

Proofs are not only geometric. For instance, next is a proof about properties of integers.

Theorem 1.2. *The sum of two even numbers is also even.*

Proof. Consider two even numbers a and b. Since they are even, then they are a multiple of two. In other words $a = 2n$ and $b = 2m$, and n and m are two whole numbers. Hence,

$$a+b = 2n+2m = 2(n+m)$$

which means that $a+b$ is a multiple of two, and thus even (that is what we wanted to show). □

In the next one we will need to have some knowledge about triangles in order to finish the proof. We will look at two proofs, using the same ideas but expressed in different ways.

Theorem 1.3. *An isosceles right triangle must have two* 45° *angles.*

Proof. Since the sum of the angles of a triangle is 180° then, the triangle already having a 90° angle, it must have two other angles adding up to 90°. Since an isosceles triangle has (at least) two angles that have the same measure then the angles that are not right must have the same measure (this is because a triangle could not have two right angles). It follows that the small angles must measure 45°. □

Proof. Let α,β and γ be the three angles of the given triangle. By hypothesis $\gamma = 90°$. Since the sum of the angles of a triangle is 180° then, $\alpha+\beta = 90°$. Now we use that an isosceles triangle has (at least) two congruent angles to get that $\alpha = \beta = 45°$. Note that we are using that the triangle could not have two right angles (see exercise 1.1). □

1.2 Contradiction

Let us suppose we want to show that a property P is true. When we prove by contradiction we will assume that P is false, and we will use logic, deductions, etc until we get to something that is impossible (things like $1 = 0$, even = odd, negative = positive, etc). Since we are reaching something clearly false then our assumption of P being false cannot be right, this forces P to be true!! Hence, we have reached our goal (proving that P is true) without using direct proof.

Let us look at a couple of examples.

Theorem 1.4. *A triangle has at most one obtuse angle.*

Proof. By contradiction. Let us assume that a triangle $\triangle ABC$ has more than one obtuse angle. Let α and β be two obtuse angles of $\triangle ABC$. Since $\alpha+\beta > 180°$, then the sum of the angles of $\triangle ABC$ is more than 180°. This is impossible!

It follows, by contradiction, that at most one angle of $\triangle ABC$ can be obtuse. □

Another example.

Theorem 1.5. *Two lines with a common perpendicular must be parallel.*

Proof. By contradiction. Assume that there are two lines, ℓ and m, that have a common perpendicular t and that are not parallel. Let C be the point of intersection of ℓ and m, and let A and B the points of intersection of ℓ and m with t, respectively.

Note that $\triangle ABC$ has two right angles (at A and B). This is impossible!! (see exercise 1.1).

By contradiction, we get that two lines having a common perpendicular must be parallel. □

A classical proof by contradiction follows.

Theorem 1.6. *$\sqrt{2}$ is not a rational number.*

Proof. By contradiction. Assume that $\sqrt{2}$ is a rational number, and thus

$$\sqrt{2} = \frac{a}{b}$$

where a and b are whole numbers with no common factors (we want the fraction to be in least terms).

By squaring and cross multiplying we get $2b^2 = a^2$. Since the number on the left is a multiple of two, then so must a be. It follows that $a = 2n$ for some whole number n. Let us plug $a = 2n$ into $2b^2 = a^2$. We get,

$$2b^2 = (2n)^2$$

which is

$$b^2 = 2n^2$$

but this forces that b is a multiple of two. Impossible!! a and b cannot be both multiples of two, as they have no common factors.

By contradiction, $\sqrt{2}$ is not a rational number. □

1.3 Constructions

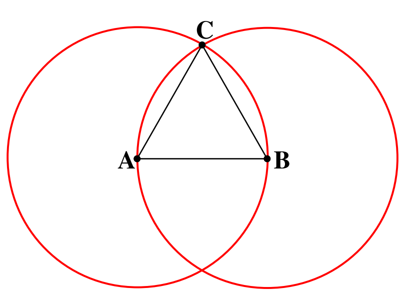
A construction of a shape, polygon, or figure with certain given properties is the base of geometry, as by doing constructions one can assure that the objects we are studying and learning about really do exist, and also discover things that had not been caught before.

Even though constructions may not be considered as proofs, the structure and general idea of not doing, or claiming, something that is not fully justified is present in both proofs and constructions.

In order to construct what is asked to us we need to follow a few simple rules, these are called postulates, and are discussed in chapter 3. For now, without really discussing what the postulates are we will look at a couple of simple constructions to illustrate how to construct shapes using a straight edge and a compass (which means we will construct shapes by drawing lines and circles only).

Example 1.1. Let us construct an equilateral triangle with a given base.

Given segment $\overline{AB}$ with length a. First draw circles with centers A and B and radius a. The intersection of these circles is called C (note we have two options for C, choose either). Draw lines to create $\triangle ABC$, which must be equilateral because of the radius considered for the circles.



Example 1.2. Assume you know how to construct a perpendicular line to a given line at a given point. We want to construct a square given one side of it.

We will just give the construction in words, as an exercise, you should take a straight edge and a compass and perform the construction with the instructions given.

Given segment $\overline{AB}$ with length a. At A and B construct perpendicular lines to $\overline{AB}$. We will use only the parts of these lines that are 'above' $\overline{AB}$. At A and B draw a circle with radius a. These circles will intersect their corresponding perpendiculars in exactly one point. Label these points C ('above' A) and D ('above' B), and join A with C, B with D, and C with D with lines. The shape $ABCD$ is a square with side a.

Example 1.3. We want to construct a $60°$ angle with vertex at a given point A. As in the previous example, you should perform the construction following the instructions given next.

Draw any line through A and a circle with any radius (call it a) centered at A. The line and the circle will intersect in two points, choose one and label it B. So far we have constructed a segment with length a. By example 1.1 we can construct an equilateral triangle with base $\overline{AB}$. Since equilateral triangles have $60°$ angles at each vertex, then we have just constructed the angle at A that we wanted to get.

Many more constructions will be shown in this book. Keep in mind that, just as in the last two examples, previously known constructions can be used to do new, more complex, constructions. Practice this skill, it is a very important one to have.

Problems

1.1. Prove that a triangle has at most one right angle.

1.2. What is the set of points that are all at the same distance from a fixed point C?

1.3. Construct a triangle with sides 2 units, 4 units and 5 units.
Hint: Use the longer segment as the base and then use circles with radii 2 and 4 to find the third vertex.

1.4. Can you construct a triangle with sides 2 units, 4 units and 7 units?

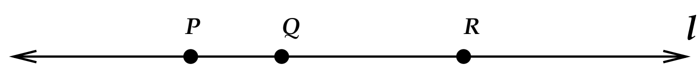
Chapter 2
Basic plane Euclidean geometry

Before getting into anything complex, in terms of geometry, we need to set what will be the objects we will use to do geometry. These concepts are probably known by you, thus their descriptions will be short and, sometimes, appealing to your intuition or previous knowledge.

We will think of a **point** as a dot on a piece of paper. A point has no length or width, it just specifies an exact location. It is interesting that even though a point is almost nothing all geometric shapes are collections of points.

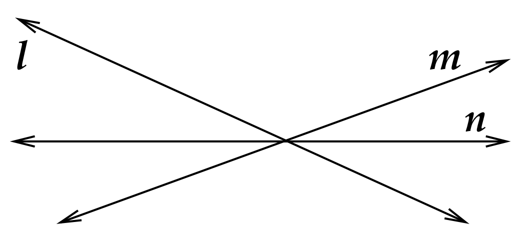
We may think of a **line** as a 'straight' line that we might draw with a ruler on a piece of paper, except that in geometry, a line extends forever in both directions. A line passing through two different points A and B is written as $\overleftrightarrow{AB}$. Note that the idea of 'line' and 'straight' are linked, and that if one uses one to define the other (as we did above) then probably we would use the other to define the one. This is not quite correct, but we will (ab)use our intuition in this definition.

Three or more points are said to be **collinear** if there is a line that contains them. In the picture, the line l contains the points P, Q and R. Hence, P, Q and R are collinear



Since, there is always a line through any two given points, then we could say that two points are always collinear.

Two lines either intersect or they are parallel. Note that this could be used as a definition of parallel lines: lines that do not intersect. Three or more lines are **concurrent** if they all pass through the same point. In the following picture, l, m and n are concurrent.



A **ray** is the portion of a line that has one endpoint and extends indefinitely from the endpoint on. A ray with endpoint A and passing through a point B is written as $\overrightarrow{AB}$ or $\overleftarrow{BA}$.

The rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are **opposite rays** if they are distinct and the points A, B and C are collinear. Opposite rays form a $180°$ angle.

A **line segment** is the portion of a line that is between two points (the two points included). A line segment with endpoints A and B is written as $\overline{AB}$. The **length** of the segment $\overline{AB}$ is the distance between A and B. Two segments, $\overline{AB}$ and $\overline{CD}$, having the same length are said to be **congruent**. We denote that as $\overline{AB} \cong \overline{CD}$.

Remark 2.1. Given two distinct points, the distance between them is always positive (in particular never zero).

A point P between A and B such that $\overline{AP} \cong \overline{PB}$ is called the **midpoint** of the segment $\overline{AB}$. The midpoint is said to **bisect** the segment.

Remark 2.2. The existence of a midpoint will be given later by finding a way to explicitly construct it from A and B. Moreover, given a segment, its midpoint is unique.

Proof (Uniqueness of a midpoint). Assume there are more than one (distinct) midpoint of $\overline{AB}$, call two of them P and Q. We know, by remark 2.1 that the distance between P and Q is not zero, but this is impossible, as both are equidistant from A and B. Contradiction. □

An angle with vertex A is a point together with two rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ (called the sides of the angle) emanating from A. We call this angle $\angle BAC$ or $\angle CAB$. Often we will use lowercase greek letters to denote angles.

It is very customary to identify an angle with its measure. Be careful with this, as the measure of an angle only captures part of what the angle really is. In fact, two angles with the same measure are said to be **congruent**.

We will mostly use the sexagesimal system to represent angles. That is, we will consider angles to have a measure between $0°$ and $360°$. However, there are other ways to measure angles. Radians are the most used in trigonometry, calculus, etc. In this system and angle of $180°$ corresponds to π radians. This correspondence defines a proportionality between these twi types of measures.

For example, if you want to know how many radians is $45°$, then by using proportions one gets

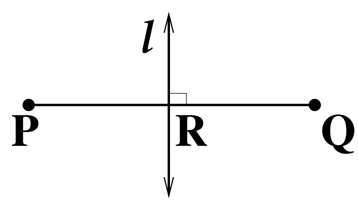
$$\frac{180°}{\pi} = \frac{45°}{x}$$

Solving for x one obtains that a $45°$-angle also measures $\frac{\pi}{4}$ radians.

Remark 2.3. Given two distinct rays with a common endpoint, the angle formed by them has always positive measure.

A **bisector** of a segment is a line that passes through the midpoint of the segment. If the bisector intersect the given line forming a $90°$ angle, then it is called a **perpendicular bisector**.

In the picture l is the perpendicular bisector of $\overline{PQ}$, and R is the midpoint of $\overline{PQ}$



Remark 2.4. There is a very simple way to construct the midpoint and/or the perpendicular bisector of a given segment $\overline{AB}$ by using just an unmarked ruler and a compass.

One first draws two circles with the same radius centered at A and B, The radius must be large enough so the circles intersect at two points. The line joining these two points is the perpendicular bisector of $\overline{AB}$, and thus passes through the midpoint of $\overline{AB}$. We will see later why this works, we first need to learn a few things about triangles.

Definition 2.1. Two angles are said to be **congruent** if one of them could be placed on top of the other for a perfect match. Congruent angles have the same measure.

Two angles with a common vertex and that share a side are said to be **adjacent angles**. Two nonadjacent angles formed by two intersecting lines are called **vertical angles**.

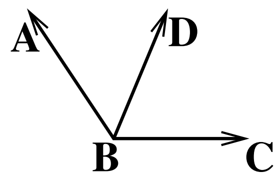
Remark 2.5. Theorem 1.1 says that two vertical angles must be congruent (Vertical Angle Theorem or VAT).

An **acute angle** is an angle whose measure is greater than $0°$ and less than $90°$. A **right angle** has measure exactly $90°$. An **obtuse angle** is an angle whose measure is greater than $90°$ and less than $180°$. A **straight angle** has measure exactly $180°$.

Two angles are called **complementary** if the sum of their measures is exactly $90°$. Two angles are called **supplementary** if the sum of their measures is exactly $180°$. Note that two complementary (or supplementary) angles need not to be adjacent. A **linear pair of angles** are adjacent angles whose non-common sides are opposite rays (form a straight line), these angles must be supplementary.

An **angle bisector** is a ray whose endpoint is the vertex of the angle and which divides the angle into two congruent angles.

In the picture, $\overrightarrow{BD}$ is a bisector of $\angle ABC$ if and only if $\angle ABD \cong \angle DBC$.



Remark 2.6. There is a nice and simple way to construct the angle bisector of a given angle.

This construction uses the one we did for the perpendicular bisector of a segment, and thus the details will be clarified later when we go over triangles.

First you draw a circle centered at the vertex of the angle, this circle intersect the sides of the angle in one point each. Call these points A and B. It turns out that the perpendicular bisector of $\overline{AB}$ goes through the vertex of the angle and, in fact, is the angle bisector we were looking for.

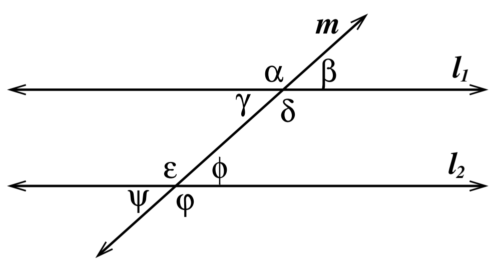
Two lines ℓ and m are perpendicular if they intersect at a point P and if there is a ray that is part of ℓ and a ray that is a part of m that form a right angle. Perpendicular lines ℓ and m are denoted by $\ell \perp m$.

Remark 2.7. The distance between a point P and a line ℓ is the length of the segment that starts at P and hits ℓ in a right angle. That is, the distance from P to ℓ is given by a perpendicular line to ℓ through P.

Two lines in the same plane which never intersect are called **parallel lines**. We say that two line segments are parallel if the lines that they lie on are parallel. If line ℓ_1 is parallel to line ℓ_2 we write $\ell_1 || \ell_2$.

Remark 2.8. If one throws a perpendicular to, let us say, ℓ_1 at ANY point $P \in \ell_1$, then the segment created 'between' the lines ℓ_1 and ℓ_2 has always the same length, independently of the point P... in other words, two parallel lines are always equidistant.

Let l_1 and l_2 be two lines and m a transversal to both of them forming eight angles, like in the following picture,



We can see that we get pairs if angles that look congruent (for example, α and ε). If any of these pairs is a pair of congruent angles then $l_1 || l_2$, and $\alpha = \varepsilon = \delta = \varphi$ and $\beta = \gamma = \phi = \psi$.

Conversely, if $l_1 || l_2$ then $\alpha = \varepsilon = \delta = \varphi$ and $\beta = \gamma = \phi = \psi$.

Problems

2.1. In the proof of remark 2.2. Explain with a picture why the distance between P and Q being not zero contradicts that both points are equidistant from A and B.

Note that you are using that the midpoint(s), A and B are collinear. Explain why this should be true.

2.2. Prove that given a segment, its perpendicular bisector is unique.
Hint: Use remark 2.3 and the idea used to prove remark 2.2.

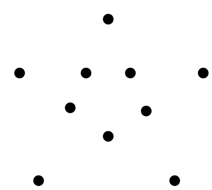
2.3. Prove that, given an angle, its angle bisector is unique.

2.4. Given 4 points in a plane. How many lines are determined? Note that considering different locations for the points will yield different answers.

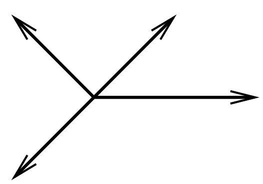
2.5. The intersection of two rays might be.....
What about their union?

2.6. Suppose K, L, and M are on a line ℓ, with L between K and M. Which term best describes the set of all points P such that L is between M and P ? (assume that L could be equal to P).

2.7. How many lines are determined by the ten points in the diagram? (Points that appear to be collinear are collinear)

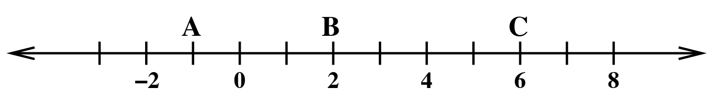


2.8. Show that there are exactly 12 positive angles less than 360° in the figure.



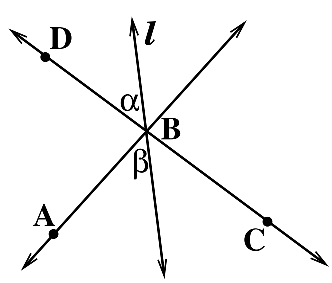
2.9. Find the measures in radians of the angles 0°, 30°, 60°, 90°, 270°, and 360°.

2.10. Find a point P such that $\overline{AP} \cong \overline{PC}$. Then find a point Q such that $\overline{AB} \cong \overline{BQ}$



2.11. An angle's measure is three times the measure of its supplement. What is the measure of the angle ? What is its complement?

2.12. Consider the following picture.



where l is the bisector of the angle $\angle ABC$. Prove that $\alpha = \beta$.

Chapter 3
Postulates and constructions

Euclid wrote his famous book, The Elements, about 23 centuries ago. His idea was to summarize all the mathematical knowledge of his times by obtaining results (theorems) from known ones, and this known ones were to be obtained also from other, more basic, known results, etc. Proceeding in this fashion, he eventually got to as set of very obvious, or self-evident, statements that would be the base of everything (math of course)!! He called these evident statements Common Notions and Postulates. Of course, before what follows, Euclid listed all necessary definitions. We will not define everything in detail, as most of the concepts discussed here are known to us.

What follows is, essentially, taken from book I of Euclid (that explains the weird way it is all written. Although I have changed the phrasing so it is all easier to understand). Visit

$$http://aleph0.clarku.edu/\sim djoyce/java/elements/elements.html$$

for a more complete review (with pictures and Java applets. Very nice stuff.)

3.1 Common Notions

These are more numerical or algebraic, but are necessary for what follows.

1. Things which equal the same thing also equal one another.
2. If equals are added to equals, then the wholes are equal.
3. If equals are subtracted from equals, then the remainders are equal.
4. Things which coincide with one another equal one another.
5. The whole is greater than the part.

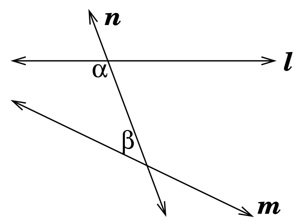
Note that we use these notions every day when we solve equations and inequalities. However, the phrasing is so vague that, for example, 'the whole' is not necessarily a number or algebraic expression, it could also be an angle, the area of a shape, the length of a segment, etc.

3.2 Postulates

Now is when the geometry starts. These postulates assume that all elements considered (points, lines, etc) are on the same plane and that all lines have infinite length.

1. It is possible to draw a straight line from any point to any point. **(Every two points determine a unique line)**.

2. It is possible to extend a segment continuously into a whole line.
3. It is possible to describe a circle with any (known) center and (known) radius.
4. All right angles equal one another. **(A right angle had been defined by Euclid as an angle that is congruent to its supplement)**.
5. If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.
   The 'translation' of this postulate says that since in the picture



   the sum of the angles α and β is less than 180°, then l and m, once extended into whole lines, will intersect 'on the side' of n where α and β are.

Since Euclid wrote The Elements there were people thinking that the fifth postulate was special, it wasn't very 'self-evident', and clearly was much more complicated than the previous ones. In fact, it is believed that Euclid himself though the fifth was odd, as he held the use of it as long as possible; the fifth postulate is used for the first time in proposition 29.

For centuries the greatest mathematical minds attempted to determine whether the fifth was really a postulate, all of them were unsuccessful until in the nineteenth century Nikolai Lobachevsky and János Bolyai (separately) proved that the fifth postulate was independent from the previous 4 postulates (it is also 'known' that Gauss had already figured this out himself, but chose not to publish his work). They did this by constructing geometries where the first 4 postulates hold but not the fifth, their examples (essentially the same) were the first examples of non-Euclidean geometries.

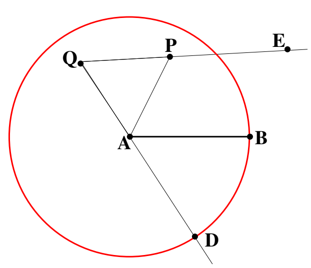
But, what is a non-Euclidean geometry? By definition it is a geometry (points and lines on a plane, and all one can construct out of them) where the fifth postulate is false, but the other four are true. We will see later a few properties of these new geometries.

Since Euclid did not use the 5^{th} postulate until proposition 29 then the first 28 propositions are valid not only in the 'standard' Euclidean geometry but also in all non-Euclidean geometry, it is customary to call Neutral Geometry to the geometry (together with all its results) that assumes the first four postulates but does not assume anything about the fifth.

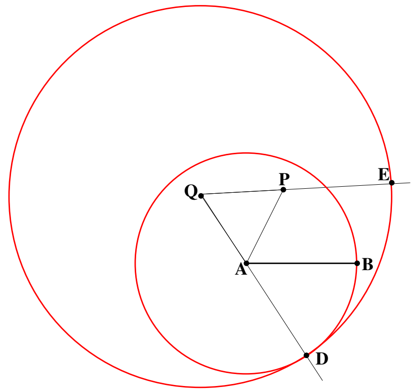
Now we will go over the arguments that prove a few of these propositions that do not need *the fifth*. These proofs/constructions are very important for you, as they are good examples of questions you might get in the constructed response part of the CSET.

1. How to construct an equilateral triangle given its base (constructed in example 1.1).
2. How to place a segment congruent to a given segment with one end at a given point. **i.e. How to copy a segment**

   Proof. Given segment $\overline{AB}$ and a point P. First join P and A with a line (postulate 1). Using proposition 1 we can find Q such that $\triangle APQ$ is equilateral. Now draw a circle centered at A with B on the boundary (postulate 3).



Extend the segments $\overline{QA}$ andn $\overline{QP}$ into lines (postulate 2), call D and E to the intersections of these lines and the circle already constructed. Now draw a circle centered at Q with D on the boundary (postulate 3).
Note that $\overline{AB} \cong \overline{AD}$ and that $\overline{QD} \cong \overline{QE}$. Since $\overline{QA} \cong \overline{QP}$, it follows that $\overline{AB} \cong \overline{AD} \cong \overline{PE}$.



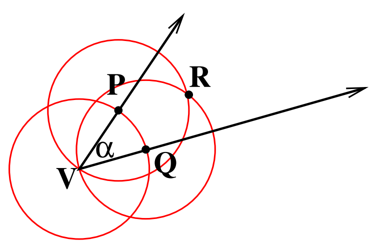
□

3. How to cut off from the larger of two given unequal segments a segment congruent to the shorter.

   Proof. Using the previous proposition we copy the shorter segment and put its beginning at one extreme of the longer segment. It follows that we can now cut off (literally) the shorter segment from the longer. □

4. *SAS* criterion for congruence of triangles (proved as theorem 6.3).
5. The base angles of an isosceles triangle are congruent (proved as theorem 6.1).
6. If in a triangle two angles are congruent to each other, then the sides opposite the equal angles are also congruent to each other (proved as theorem 6.1).
   Note that this is the converse of the previous proposition.
8. *SSS* criterion for congruence of triangles (proved as theorem 6.4).
9. How to bisect a given angle α.

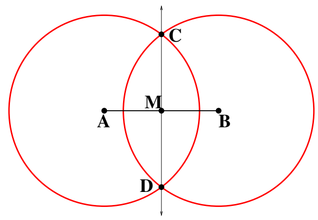
   Proof. We first draw a circle centered at the vertex V of α.



   The intersections of this circle with the sides of the angle are labeled P and Q. Now we draw two circles **with the same radius** centered at P and Q. These circles intersect at R (note we have two choices for R, choose either).
   Note that the triangles $\triangle VPR$ and $\triangle VQR$ are congruent by *SSS*. It follows that $\angle PVR \cong \angle QVR$. Thus $\overrightarrow{VR}$ is the bisector of α. □

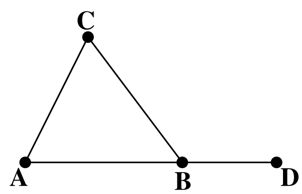
10. How to bisect a given segment.

    Proof. The segment $\overline{AB}$ is given. We draw circles **with the same radius** centered at A and B. These circles intersect in two points, which we label C and D.
    The line through C and D intersects $\overline{AB}$ at a point M. We claim that M is the midpoint of $\overline{AB}$. Moreover, that the line $\overleftrightarrow{CD}$ is the perpendicular bisector of $\overline{AB}$.
    The claim follows from the fact that the points C and D are equidistant from A and B and thus $\triangle CBD \cong \triangle CAD$. It follows that $\angle BCD \cong \triangle ACD$, which implies that $\triangle CMB \cong \triangle CMA$ by *SAS*. Hence $\overline{AM} \cong \overline{MB}$.
    The fact that $\overleftrightarrow{CD}$ is the perpendicular bisector of $\overline{AB}$ follows from the fact that the angles $\angle AMC$ is congruent to $\angle CMB$ and that their sum is $180°$. □

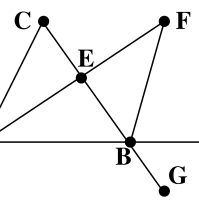


11. How to draw a line at right angles to a given line from a given point on it.
12. How to draw a line perpendicular to a given line from a given point not on it.
15. If two straight lines cut one another, then they make the vertical angles equal to one another.
    This is the vertical angle theorem. It was proved as theorem 1.1.
16. In any triangle, if one of the sides is extended, then the exterior angle is greater than either of the interior and opposite angles.

    Proof. We want to show that in the picture below $\angle CBD$ is larger than both $\angle CAB$ and $\angle ACB$.

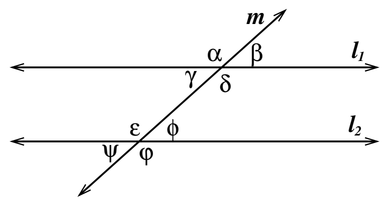
    Using proposition 10 we find the midpoint of $\overline{CB}$, call it E. Then we draw the line from A to E, and using proposition 2, and postulate 2, we find a point F on it such that $\overline{AE} \cong \overline{EF}$. We obtain the picture

    where E is the midpoint of both $\overline{CB}$ and $\overline{AF}$. It follows that $\triangle AEC \cong \triangle FEB$ by SAS, and thus $\angle ACB \cong \angle FBE$, which is smaller than $\angle EBD$.
    Similarly, we obtain that $\angle CAB$ is smaller than $\angle EBD$. □

    Note that in the proof above, we need to find the point F by extending a line until it reaches a desired length. This is done by assuming that lines extend indefinitely. Hence, for this proposition to work we need infinite lines.
26. *ASA* criterion for congruence of triangles (proved as theorem 6.5).
    Note that if we know that two triangles have two pairs of corresponding congruent angles, then the third angle of one triangle must be congruent to the third angle of the other. Hence, *ASA* can be also phrased as *AAS* or *SAA*.
27. If a line is transversal to two lines making the alternate angles congruent to one another, then the straight lines are parallel to one another.
28. If a line is transversal to two lines making the exterior angle congruent to the interior and opposite angle on the same side, or the sum of the interior angles on the same side equal to 180°, then the lines are parallel to one another.

    The previous two propositions say that, in the following picture, if $\gamma = \psi$ or $\gamma = \phi$, then l_1 is parellel to l_2



As mentioned before, all the results above do not require the fifth postulate. But there are many, many, MANY others that do. In fact, most of what we will see in this course depends in one way or another on this postulate.

3.3 Non-Euclidean geometry

In order to see what we obtain when we decide to assume that the fifth postulate is false, we need to look at other better-known results that turn out being equivalent to the fifth. Here there are a few

1. (Playfair's axiom) Let ℓ be a line and P a point not on ℓ, then there is exactly one line passing through P that is parallel to ℓ.
2. The sum of the interior angles of a triangle is exactly 180°.
3. (Proposition 29) If in figure used for propositions 27 and 28 we assume $l_1||l_2$, then the alternate interior angles are equal, and the corresponding angles are congruent.
4. The area of a triangle is half its base times its height.

Now that we are more familiar with what the fifth means we can see what it means to assume it is false. We will focus mostly on Playfair's axiom, as the negation of it gives us two options

(i) Let ℓ be a line and P a point not on ℓ, then there are more than one line passing through P that are parallel to ℓ. In this case we obtain what is called *hyperbolic geometry*. An example of this is the hyperbolic plane (its description is too complex this course).

(ii) Let ℓ be a line and P a point not on ℓ, then there are no lines passing through P that are parallel to ℓ
In this case we obtain what is called *elliptic geometry*. An (almost) example of this is the geometry of the sphere, where the lines are the great circles (meridians, equators, etc).

The following table shows how things work in the three geometries.

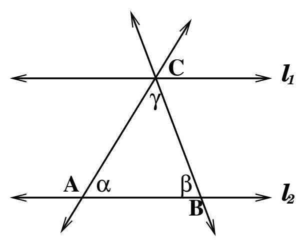
	Euclidean	Hyperbolic	Elliptic
The Fifth	True	False	False
Proposition 29	True	False	False
Number of parallel lines to ℓ through $P \notin \ell$	1	∞	0
Angle-sum of a triangle	180°	$< 180^\circ$	$> 180^\circ$
Area of a triangle	$\frac{b \cdot h}{2}$	It depends on the angle-sum	It depends on the angle-sum

Problems

3.1. Prove propositions 11 and 12.
Hint: Use the idea used in the proof of remark 2.2 after obtaining an interval on the given line using the point given.

3.2. Assume the angle sum of any triangle is $180°$
(a) Show proposition 27.
(b) Show proposition 28.

3.3. In this exercise you will prove that the angle sum of any triangle is $180°$
(a) Consider a triangle $\triangle ABC$ and use Playfair's axiom to get a line through C that is parallel to $\overleftrightarrow{AB}$.
(b) Extend the three sides of the triangle to obtain the picture



Use proposition 29 to determine the angles at C, and then get the desired result.

3.4. Choose two points on a sphere/ball. Draw the shortest path on the sphere that joins these points. Convince yourself that this 'line' must be a great circle of the sphere.

3.5. Draw a triangle on a sphere/ball. Note that on the sphere lines are 'curvy', and that triangles formed with these lines are like a swollen (Euclidean) triangle. Use a protractor to measure the angle-sum of this triangle. It should be more than $180°$, is it?

Repeat this exercise with many different triangles on the sphere. Do you note any relation between the angle-sum of the triangles and their area?

3.6. Show that (in Euclidean geometry) the sum of two angles of a triangle is equal to the opposite exterior angle of the triangle. Show this is not true in non-Euclidean geometry.

3.7. A man comes out of his house, walks 10 miles South, then 10 miles East, and finally 10 miles North. He is now back at his place. When he is getting inside he sees a bear. What color is the bear?

3.8. If the sum of the interior angles of a triangle is equal to $200°$ then how many parallel lines to a line ℓ can be found through a point P not on ℓ?

What if the sum of the interior angles of a triangle were $100°$?

Chapter 4
Congruency and similarity

When we have two figures/shapes that overlap perfectly once one of them is placed on top of the other we will say that the shapes are congruent. The symbol for 'congruent to' is $\cong$. The idea behind this concept is that there are movements of the plane that preserve shapes, angles, distances, etc. These movements will be discussed later when we learn about transformations in chapter 12.

In case there is a movement of the plane that preserves shapes and angles but not distances then these movements are essentially 'zooming' (in or out) one shape into the other. Two figures that are related this way are said to be similar. The symbol for 'similar to' is $\sim$.

We can make these concepts more precise for polygons and other familiar shapes.

Definition 4.1. Two given polygons are said to be congruent if by creating a one-to-one correspondence between their vertices then we get that all pairs of corresponding angles and all pairs of corresponding sides are congruent.

Similarly, in order to get a definition of similarity of polygons we need the fact that when we zoom in/out (to check similarity) we use the same scaling for all distances on the plane, the scale used can be obtained by dividing the lengths of corresponding sides. That is, by the ratio of two corresponding sides.

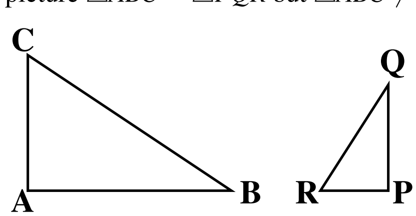
Definition 4.2. Two polygons are said to be similar if there is a one-to-one correspondence between their vertices such that all pairs of corresponding angles are congruent and the ratios of the measures of all pairs of corresponding sides are equal.

Remark 4.1. If two shapes are congruent then they are similar. Two similar shapes need not be congruent.

In the following proposition we summarize a few easy results about congruency and similarity,

Proposition 4.1. *1. Two segments are congruent if and only if they have the same length.*
2. Any two segments are similar. The ratio is given by dividing the length of one by the length of the other.
3. Two angles are congruent if they have the same measure.
4. It two angles are similar then they are congruent (similarity preserves angles).
5. Two circles are congruent if they have the same radius.
6. Any two circles are similar. The ratio is given by dividing the radius of one by the radius of the other.

The correspondence of vertices mentioned in definitions 4.1 and 4.2 is very important, as with it we show in which way we are looking at the polygons that are congruent/similar. Mislabeling vertices could yield to wrong results. For instance, in the following picture $\triangle ABC \sim \triangle PQR$ but $\triangle ABC \nsim \triangle RPQ$.



One important thing to remark about congruency and similarity is that they are transitive. That is, if A, B and C are figures/shapes such that $A \cong B$ and $B \cong C$, then $A \cong C$. Similarly, if $A \sim B$ and $B \sim C$, then $A \sim C$.

One way to think about similarity is that two shapes are similar if one can be zoomed into the other. But how does this work? We are used to say things as 'zoom $\times 2$' but what does that mean? What is it doubled? Lengths !! For example, a square of side 3 *in* when zoomed $\times 2$ becomes a square of side 6 *in*. Note that the area of the square does NOT double but actually quadruples ($4 = 2^2$). Similarly, a cube of side 3 *in* when zoomed $\times 2$ becomes a cube of side 6 *in*. Note that the volume of the cube is multiplied times $8 = 2^3$.

Since there is not much to do so in general (for ALL shapes, or ALL polygons), then we will move on to other sections, where we will discuss congruency or similarity for specific shapes.

Problems

4.1. I want to build a pool in my yard. It must have the same shape as the yard itself but (of course) smaller. In fact, I want the perimeter of the pool and the perimeter of the yard to be in a ratio of 1 : 2. What would be the ratio of the areas enclosed by the pool and the yard?

4.2. Prove proposition 4.1.

4.3. True or False? If true, prove it. If false, give an example that supports your claim.
(a) Same area squares are similar.
(b) Same area squares are congruent.
(c) Same area rectangles are similar.
(d) Same area rectangles are congruent.
(e) Same area circles are similar.
(f) Same area circles are congruent.

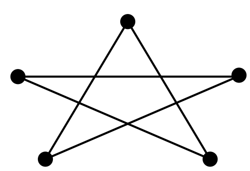
Chapter 5
Polygons

Polygons are the most basic closed shapes in geometry. They can be large/complex enough to be interesting but since they have only segments (no curves) by sides then they are fairly easy to study. Probably the most important polygon is the triangle, as it will be present in most of the techniques we will discuss to learn about polygons. Due to their importance, triangles have their own chapter (chapter 6).

5.1 General properties of polygons

Definition 5.1. A **polygon** is a set/shape formed by points (called vertices) and segments (called sides) such that every vertex is the endpoint of exactly two sides.

Note that the previous definition implies that a polygon is a closed shape. Also, under that definition a shape like

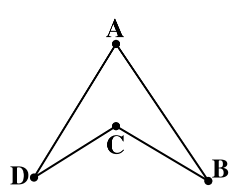


is considered to be a polygon (in this case with 5 vertices and 5 sides). However, this is not what comes to mind when we think of a polygon, as the star above has sides intersecting in points that are not vertices, we do not want this kind of behavior in the polygons we want to study.

Definition 5.2. A polygon with sides intersecting only at vertices is called a simple polygon.

Remark 5.1. In this course we will always assume polygons are simple.

Now, let us consider the following simple polygon



which, then again, might not be what we usually consider as a polygon. What we do not line about the shape above is that it is not convex (that interior angle $\angle DCB$ is too large!!).

Definition 5.3. A simple polygon is said to be convex if the segment joining any two of its points is completely contained in the 'frame' given by the polygon.

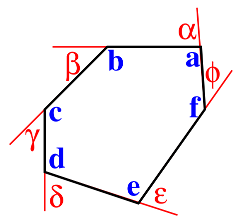
Another way to think about a convex polygon is as a simple polygon that has interior angles measuring at most 180°.

From now, in this book, all polygons are convex simple polygons.

Probably the first way to characterize a polygon is by counting how many sides it has. Although this is not enough to say much about the polygon, it is something that helps the visualization of the shape we want to study.

Definition 5.4. (i) When we say 'side $\overline{AB}$' we will mean the side that joins vertices A and B. Note that this implies that once all the vertices of the polygon are labeled, then there is a unique way to label all its sided.
(ii) A polygon having n sides is sometimes called an n-gon.
(iii) The **perimeter** of a polygon is the sum of the lengths of all the sides of the polygon.

The second feature that is used to characterize a polygon is its angles, they come in two flavors: the **interior angles** of a polygon are the angles (inside the polygon) formed by two consecutive sides of the polygon. The **exterior angles** of a polygon are the angles obtained by extending one side (either) at all vertices.



In the picture above, a, b, c, d, e, and f are internal angles, and $\alpha, \beta, \gamma, \delta, \varepsilon$ and ϕ are external angles.

Proposition 5.1. *The sum of the interior angles of a polygon with n sides equals $n \cdot 180^\circ$ minus the sum of the exterior angles.*

Proof. Since the exterior angles are obtained by extending a side, then an exterior angle is always the supplement of an interior angle (and adjacent to that angle as well). Since there are n interior angles and for each one of them we get a linear pair of angles formed by one interior angle and one exterior angle, then the sum of the measures of all angles (interior and exterior) must be $n \cdot 180^\circ$.

Separating interior and exterior angles we get

$$\text{(sum of interior angles)} + \text{(sum of exterior angles)} = n \cdot 180^\circ$$

Subtracting yields what the proposition claims. □

A diagonal is obtained by joining two distinct (and not consecutive) vertices in a polygon. We can see that a triangle has no diagonals, and that a quadrilateral has exactly two. The following result generalizes these numbers to any (simple convex) polygon.

Proposition 5.2. *The number of diagonals in an n-gon is* $\dfrac{n(n-3)}{2}$.

Proof. We count the number of diagonals by fixing a vertex (we have n to choose from), then we multiply this by the number of options for the second extreme of the diagonal (there are $n-3$ available). Thus we get $n(n-3)$ diagonals. But since the same two extremes determine the same diagonal, then we need to divide by two to get the correct number. □

Since polygons can have any number of sides, and each side be of, pretty much, any length, then it is impossible to get a through and general study of all polygons. Hence, we will focus on polygons that are more 'predictable', with have more symmetry in them.

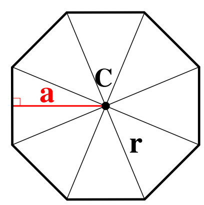
5.2 Regular polygons

Every triangle that has three angles having the same measure must have three sides with the same length. This is not true in polygons with more than three sides. For instance, a rectangle (that is not a square) has four 90° angles but it does not have four sides with the same length. Similarly, a rhombus (see chapter 7) has four sides with the same length but its angles do not have all the same measure.

Definition 5.5. An **equilateral polygon** is a polygon with all sides congruent to each other.
An **equiangular polygon** is a polygon with all interior angles congruent to each other.
If a polygon is equilateral and equiangular then the polygon is **regular**.

Example 5.1. As mentioned above. A rectangle that is not a square is equiangular but not equilateral (and thus not regular). A rhombus that is not a square is equilateral but not equiangular (and thus not regular). A triangle that is equiangular must be regular, and a triangle that is equilateral must be regular.

A regular polygon possesses a point C (called the center of the polygon) that is equidistant from all vertices of the polygon. The distance r is called the radius of the polygon. The height of the triangles formed by using the center is called the apothem of the polygon (denoted ρ or just a).



In fact, the triangles formed by throwing all the radii are isosceles and congruent to each other.

Using the idea shown in the picture above, we see that a regular n-gon can be broken into n congruent isosceles triangles. If we add the angles in these triangles we get $n \cdot 180°$, but since by counting all the angles in the triangles we also count the angles that are formed around the center of the polygon (which add up to 360°), then we get that the sum of the interior angles of the polygon is

$$n \cdot 180° - 360° = (n-2)180°$$

This idea may be generalized to any polygon, and this would imply that the sum of the exterior angles is 360°.

But if the polygon is regular we can go further, now we can learn that the measure of each interior angle is

$$\frac{(n-2)180°}{n}$$

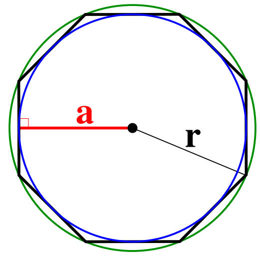
Moreover, the measure of each exterior angle of a regular n-gon is

$$180° - \frac{(n-2)180°}{n} = \frac{360°}{n}$$

which is not so surprising, as the sum of the n exterior angles of the polygon is 360°.

Remark 5.2. In the picture below it is shown that the apothem is the radius of the circle inscribed in the polygon, this circle is tangent to each side of the polygon at its midpoint. Also, the radius of the polygon is the radius of the circle circumscribed to the polygon, this circle touches the polygon only at its vertices.

Note that the more sides a regular polygon has, the closest it is to be a circle, and thus a and r to be the same number. Archimedes used this idea to find good approximations for the value of π, among many other things.



Recall that the area of a triangle is $\frac{b \cdot h}{2}$, where b is the base of the triangle and h its height. In the case of a regular n-gon, with side s and apothem a, we see that the area of the polygon is n times the area of one of the isosceles triangles formed by the radii. It follows that the area of the n-gon is

$$A = n \cdot \frac{s \cdot a}{2} = a \cdot \frac{s \cdot n}{2} = \frac{a \cdot P}{2}$$

where P is the perimeter of the polygon.

The previous formula is great to compute the area of a regular polygon if you know its perimeter (or just the length of one of its sides), but the problem is that we do not know any way to find the apothem of a regular polygon (unless it is given to us, which does not happen often).

Proposition 5.3. *If α is an interior angle of a regular polygon and s is the length of a side of the polygon, then the apothem a is given by*

$$a = \frac{s}{2} \tan\left(\frac{\alpha}{2}\right)$$

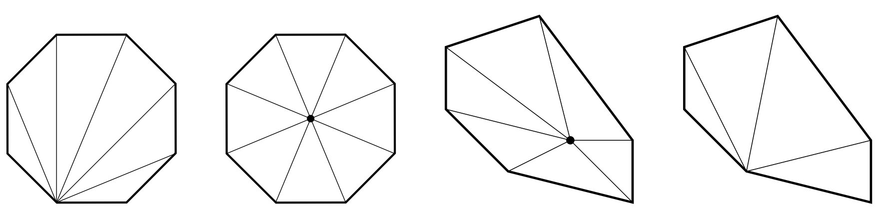
The function 'tan' *will be introduced in chapter 6, in the trigonometry section. So far, you can use your scientific calculator to compute values for* tan *if needed.*

Note that now we can find all the important features of a regular polygon, as long as we know its number of sides and the length of a side. Moreover, if one considers two n-gons with sides s and t, then these two polygons are similar, in a ratio $s : t$, and their apothems and radii are in the same ratio.

Problems

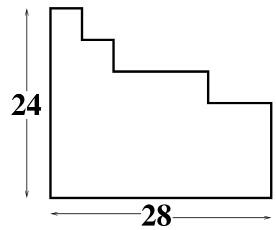
5.1. Show that the sum of the interior angles of an n-gon equals $n \cdot 180°$ minus the sum of the exterior angles of the polygon.

5.2. Assume that we know a polygon can be partitioned into triangles that share one vertex, look at the pictures below. Use this to find what the sum of the interior angles of the polygon is.



5.3. If in a regular polygon you double the length of the sides, then what happens with the perimeter and area of the polygon?

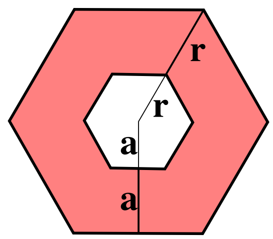
5.4. What is the perimeter of the figure below?



(All angles are right)

5.5. What is the degree measure of an interior angle in a regular octagon? An exterior angle?

5.6. In the picture below, both hexagons are regular and have the same center. Also, the big one has apothem twice the small one's. Find the area of the shaded region.



5.7. The measure of an interior angle in a regular polygon is 120°. Find the number of sides of the polygon.

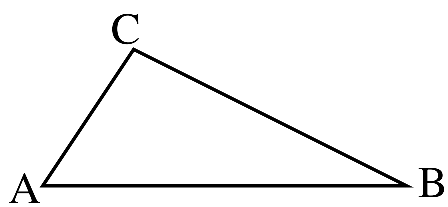
5.8. Show that the six triangles formed by the radii and the sides of a regular hexagon are equilateral.

5.9. How many sides does a regular polygon have if the measure of an interior angle is three times the measure of an exterior angle?

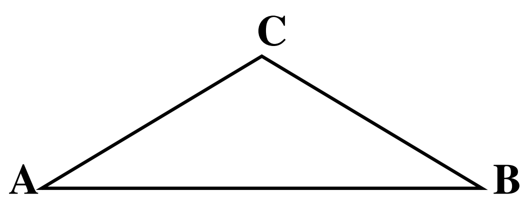
Chapter 6
Triangles

6.1 Basic properties of triangles

1. A **triangle** is a polygon with exactly three sides. Depending on the picture one can determine what the base of the triangle is. For instance, the triangle $\triangle ABC$ below



has **base** $\overline{AB}$. Also, the angles $\angle CAB$ and $\angle ABC$ are called **base angles**.
2. A triangle with at least two sides congruent is called an isosceles triangle. As a convention, we will represent an isosceles triangle as



where $\overline{AC} \cong \overline{BC}$.

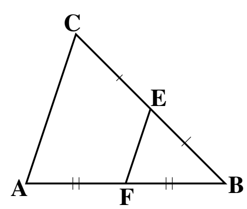
Theorem 6.1. *The base angles of an isosceles triangle are congruent. And, conversely, a triangle having congruent base angles is isosceles (these are propositions* 5 *and* 6*).*
Also, the altitude of an isosceles triangle bisects the base in a right angle, and it is also the angle bisector of the angle opposite to the base.

Proof. Call P the point of intersection of the angle bisector of $\angle ACB$ and $\overline{AB}$. This creates two triangles, $\triangle APC$ and $\triangle BPC$. Since Since $\overline{AC} \cong \overline{BC}$, $\angle ACP \cong \angle BCP$, and $\overline{PC}$ is common to both triangles, then $\triangle APC \cong \triangle BPC$ (by SAS, which we will prove later). It follows that the base angles are congruent, that $\angle APC \cong \angle BPC$, and thus that $\overline{PC}$ is perpendicular to the base of the triangle.
Moreover, if $\triangle ABC$ above has congruent base angles (but we don't know yet it is isosceles), then calling Q to the point where the angle bisector of $\angle ACB$ meets the base. Then we get two triangles $\triangle AQC$ and $\triangle BQC$, which are congruent by ASA (which we will prove later)... as $\angle ACQ \cong \angle BCQ$, $\angle CAQ \cong \angle CBQ$, and thus $\angle CQA \cong \angle CQB$ and the segment $\overline{QC}$ is common to both triangles. It follows that $\overline{AC} \cong \overline{BC}$, and thus the triangle would be isosceles. □

Remark 6.1. The previous proof can be modified to show that the points on the perpendicular bisector of a segment $\overline{AB}$ are equidistant from A and B. In fact the perpendicular bisector can be re-defined using this property, thus the perpendicular bisector of $\overline{AB}$ is the set of all points equidistant from A and B.

3. All interior angles of an equilateral triangle are congruent. That is, and equilateral triangle is a regular 3-gon. This follows from the fact that an equilateral triangle is also isosceles.
4. A triangle with a right angle is called a **right triangle**. The two sides that form the right angle are called **arms** or **legs** of the triangle, the third side is called the **hypothenuse**.
5. A triangle with one obtuse angle is called an **obtuse triangle**.
6. A triangle with three acute angles is called an **acute triangle**.
7. A **median** of a triangle is a segment from a vertex to a midpoint of its opposite side.
8. An **altitude** of a triangle is a segment starting at a vertex and ending at the closest point on the line containing the opposite side. Altitudes are always perpendicular to the line containing the opposite side.
9. The segment that joins the midpoints of two sides of a triangle is called a **midline** of the triangle.

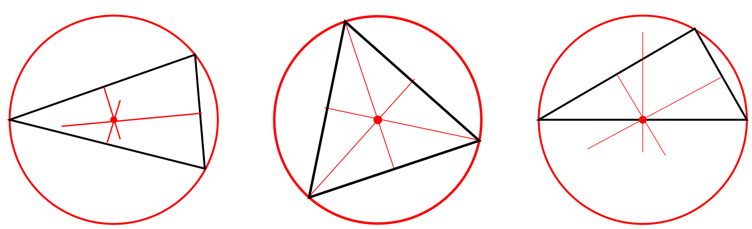
Remark 6.2. A midline's length is one half of the length of the third side of the triangle. Also, a midline is parallel to the third side. That is, in the picture



the length of $\overline{AC}$ is twice the length of $\overline{EF}$, and $\overline{AC}||\overline{EF}$.
We will soon see that $\triangle ABC \sim \triangle FBE$.

10. The **perpendicular bisectors** of a triangle are the three perpendicular bisectors of the sides of the triangle.

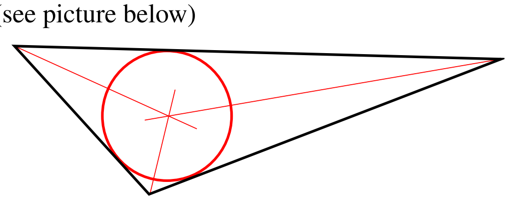
Remark 6.3. The perpendicular bisectors are concurrent, the intersection point (called the **circumcenter**) is the center of the circle circumscribed to the triangle (see pictures below)



Why is this true? Note that two perpendicular bisectors always intersect, take this intersection point and call it C. Using remark 6.1. we get that this point is equidistant to the three vertices of the triangle, which means that C is the center of a circle going through the three vertices of the triangle.
Finally, to see that the third perpendicular bisector also goes through C we just need to see that the line that goes through C and the midpoint of the side we haven't used yet contains (at least) two points that are equidistant to the vertices of the side, thus using the paragraph above we get that this line (through C and the midpoint) is the third perpendicular bisector we were missing.

11. The **angle bisectors** of a triangle are the three bisectors of the angles of the triangle.

Remark 6.4. The angle bisectors are concurrent, the intersection point (called the **incenter**) is the center of the circle inscribed to the triangle (see picture below)



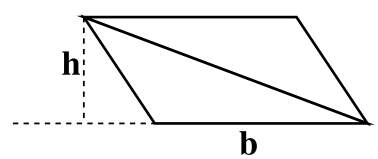
The proof to the previous remark is similar to the proof of the existence of the circumcenter. You should be able to do it.

6.2 Triangles and areas

1. **The Pythagorean theorem:** In a right triangle with legs measuring a and b and hypothenuse measuring c, then $a^2 + b^2 = c^2$
   There are hundreds of proofs for this theorem, read this for example

   $$http://mathforum.org/library/drmath/view/62539.html$$

   A couple of them are exercises in problem list 2. You should know them well. Also, read the file about this that is posted on the course's website (The 2500-year old Pythagorean theorem).
2. The area of a triangle with base b and altitude h is $A(\triangle) = \frac{bh}{2}$.
   The proof for this follows from the fact that once a triangle is given we can 'double' it to create a parallelogram

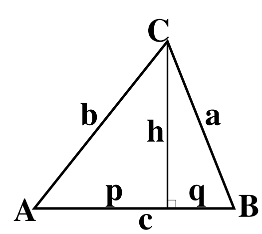
   

   Since the area of the rectangle is bh then the area of the triangle is half of that.
   Note that 'doubling' the triangle to create a parallelogram means that the two triangles forming the parallelogram are congruent. Also, we will show later that the area of that parallelogram is actually base times height.
3. Another way to find the area of a triangle is what is called Heron's formula:

   Theorem 6.2. *The area of a triangle with sides with length a, b, and c is given by*

   $$A(\triangle) = \sqrt{s(s-a)(s-b)(s-c)} \qquad \textit{where} \qquad s = \frac{a+b+c}{2}$$

   Proof. We first rotate the triangle (if necessary) so the point where the altitude intersects the base partitions it into two segments, p and q, as shown in the picture below.

   We use the Pythagorean theorem in the two triangles created by the altitude to get

   $$h^2 + p^2 = b^2 \qquad \text{and} \qquad h^2 + q^2 = a^2$$

   Since $p + q = c$ then

   $$\begin{aligned} q^2 &= (c-p)^2 \\ &= c^2 - 2cp + p^2 \\ a^2 - h^2 &= c^2 - 2cp + p^2 \end{aligned}$$

   which implies (using $h^2 + p^2 = b^2$) that

$$a^2 = c^2 - 2cp + b^2 \qquad \text{and thus} \qquad p = \frac{b^2 + c^2 - a^2}{2c}$$

Using that we get,

$$\begin{aligned}
c^2h^2 &= c^2(b^2 - p^2) \\
&= c^2(b-p)(b+p) \\
&= c^2\left(b - \frac{b^2+c^2-a^2}{2c}\right)\left(b + \frac{b^2+c^2-a^2}{2c}\right) \\
&= c^2\left(\frac{2cb - (b^2+c^2-a^2)}{2c}\right)\left(\frac{2cb + (b^2+c^2-a^2)}{2c}\right) \\
&= \frac{1}{4}\left(a^2 - (b^2 - 2cb + c^2)\right)\left((b^2+2cb+c^2) - a^2\right) \\
&= \frac{1}{4}\left(a^2 - (b-c)^2\right)\left((b^2+c)^2 - a^2\right) \\
&= \frac{1}{4}(a+b-c)(a-b+c)(b+c+a)(b+c-a)
\end{aligned}$$

We now use that

$$a+b-c = 2(s-c) \qquad a-b+c = 2(s-b) \qquad b+c+a = 2s \qquad b+c-a = 2(s-a)$$

and that the square of area of the triangle is $A = \frac{h^2c^2}{4}$ to get the formula we wanted. □

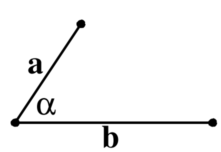
6.3 Congruency of triangles

Recall that (chapter 4) two triangles are congruent if there is a correspondence between the vertices of two triangles that yields congruent correspondent sides and congruent corresponding angles. So, in theory, in order to check that two triangles are congruent we need to check six congruences (three for sides, and three for angles), that is too much to check. Luckily, there are congruence criteria for triangles that only ask for three things to check.

Theorem 6.3 (SAS). *If two triangles have two sides congruent to two sides respectively, and have the angle contained by the congruent sides congruent, then the triangles are congruent.*

Proof. The proof for this is by constructing a triangle with the given information, and realizing that there could be only one such a triangle.

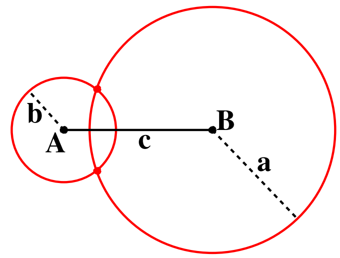
Assume that two lengths, a and b, and the angle α between them are given, thus we have a picture like.



It is pretty clear that there is a unique way to complete that picture to create a triangle. Done.

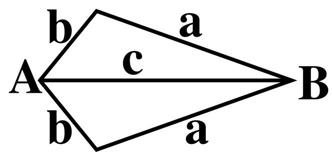
Theorem 6.4 (SSS). *If a triangles has its three sides congruent to the sides of a second triangle, then the triangles are congruent.*

Proof. This proof is similar to the previous one. We assume three lengths are given: a, b, and c. We set c as the base of the triangle (with extremes A and B), and then we draw circles centered at A and B with radii a and b. We get the following picture.



We note that the two points of intersection of the circles are both at distance b from A and a from B. Thus these two points are the only candidates to be the third vertex of the triangle we want to get.

Now I will just say that these two triangles are congruent (finishing the proof), as one of them is the reflection of the other

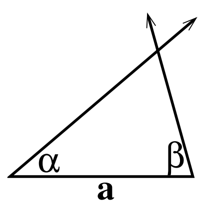


... however, it is not that easy to prove that.

Theorem 6.5 (ASA). *If two triangles have two angles congruent to two angles respectively, and one side equal to one side, namely, either the side adjoining the equal angles, or that opposite one of the equal angles, then the triangles are congruent.* ***Note that is a little more than just*** *ASA.*

Proof. First of all we realize that since knowing two angles in a triangle immediately tells us what the third angle MUST be, then it is enough to show ASA.

So, we assume we know two angles α and β and their common segment a. This yields the following picture.

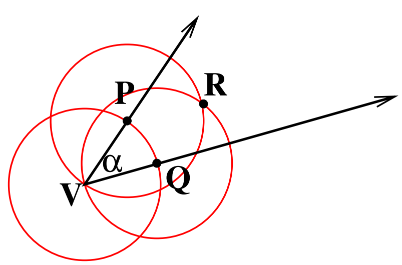


which clearly show that there is a unique way to obtain a triangle with that information.

Now a couple of classical constructions that use the latest three results. We sort of discussed these before but without so much detail.

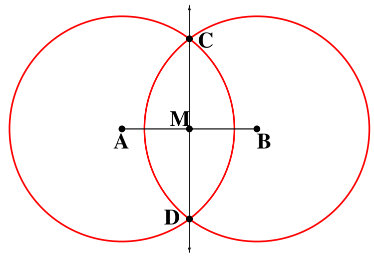
I How to bisect a given angle α.

Proof. We first draw a circle centered at the vertex V of α. The intersections of this circle with the sides of the angle are labeled P and Q. Now we draw two circles **with the same radius** centered at P and Q. These circles intersect at R (note we have two choices for R, choose either). Note that the triangles $\triangle VPR$ and $\triangle VQR$ are congruent by SSS. It follows that $\angle PVR \cong \angle QVR$. Thus $\overrightarrow{VR}$ is the bisector of α



II How to bisect a given segment.

Proof. The segment $\overline{AB}$ is given. We draw circles **with the same radius** centered at A and B. These circles intersect in two points, which we label C and D. The line through C and D intersects $\overline{AB}$ at a point M. We claim that M is the midpoint of $\overline{AB}$. Moreover, that the line $\overleftrightarrow{CD}$ is the perpendicular bisector of $\overline{AB}$.



The claim follows from the fact that the points C and D are equidistant from A and B and thus $\triangle CBD \cong \triangle CAD$. It follows that $\angle BCD \cong \triangle ACD$, which implies that $\triangle CMB \cong \triangle CMA$ by SAS. Hence $\overline{AM} \cong \overline{MB}$.
The fact that $\overleftrightarrow{CD}$ is the perpendicular bisector of $\overline{AB}$ follows from the fact that the angles $\angle AMC$ is congruent to $\angle CMB$ and that their sum is $180°$.

6.4 Similarity of triangles and trigonometry

1. The definition of similarity tells us that if $\triangle ABC \sim \triangle A'B'C'$ then

$$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}$$

where AB means the length of $\overline{AB}$ (similar for the others)... and that all corresponding angles are congruent, which means that

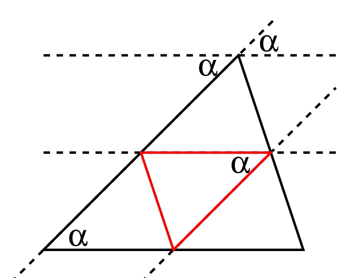
$$\angle ABC \cong \angle A'B'C' \qquad \angle BCA \cong \angle B'C'A' \qquad \angle CAB \cong \angle C'A'B'$$

Recall that $\triangle ABC \sim \triangle A'B'C'$ and $\triangle CAB \sim \triangle A'B'C'$ might mean different things because the correspondence of vertices/sides/angles is determined by the way things are written.
Since triangles are fairly simple, we can find criteria for similarity, just as we did with congruency of triangles. We have two main results:

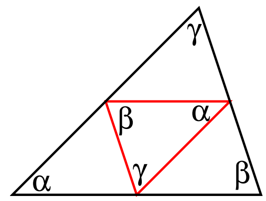
i If two triangles have two pairs of corresponding angles that are congruent, then they are similar (also called AA).
ii If two corresponding sides are in the same ratio and the angles they form (in each triangle) are congruent, then the triangles are similar (this is some sort of SAS with ratios).

Remark 6.5. The triangle formed by the midlines of a triangle is similar to the original triangle.

This is true because once parallel lines are thrown, and the sides of the smaller triangle are extended we get

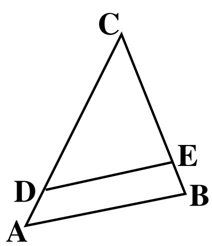


which creates a correspondence between vertices of the big and small triangles. The same can be done for all the other angles. One gets,

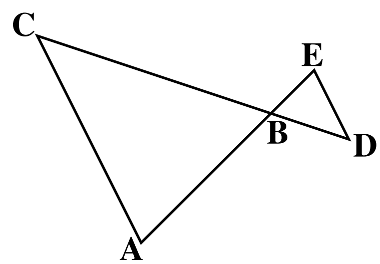


So, the triangles are similar by AA.

Remark 6.6. The 'triangle inside another triangle' and the 'bowtie' figure are classical cases of similar triangles. In fact, if

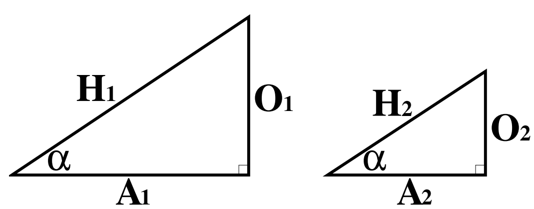


where $\overline{AB}||\overline{DE}$, then $\triangle ABC \sim \triangle DEC$. And if



where $\overline{AC}||\overline{DE}$, then $\triangle ABC \sim \triangle EBD$.

2. Now we will look at similarity of right triangles. Note that since two right triangles have both a right angle, then as soon as they have a second angle 'in common' (meaning congruent) then they will be similar. Consider the following two similar right triangles,



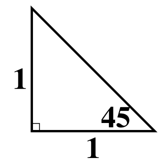
where the A's indicate 'adjacent to α', the O's denote 'opposite to α', and the H's denote 'hypothenuse'. By similarity we get

$$\frac{A_1}{H_1} = \frac{A_2}{H_2} \qquad \frac{O_1}{H_1} = \frac{O_2}{H_2} \qquad \frac{O_1}{A_1} = \frac{O_2}{A_2}$$

Since these ratios depend only on the angle α, then they are functions of α, we give them the following names

$$\sin\alpha = \frac{Opp}{Hyp} \qquad \cos\alpha = \frac{Adj}{Hyp} \qquad \tan\alpha = \frac{Opp}{Adj}$$

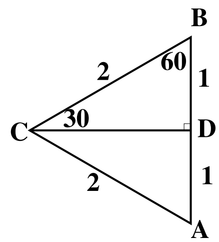
Let us find the sine and cosine of some important angles. For instance, using the isosceles triangle



we can find the third side to be $\sqrt{2}$ by using the Pythagorean theorem, then we can compute

$$\sin 45^o = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \qquad \text{and} \qquad \cos 45^o = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Now consider a right triangle $\triangle ABC$ that is equilateral and with $\overline{CD}$ to be one of its altitudes/perpendicular bisectors/angle bisectors like in the picture below



Using the Pythagorean theorem we get that the altitude's length is $\sqrt{3}$. Now looking at the triangle $\triangle CDB$ we can find that

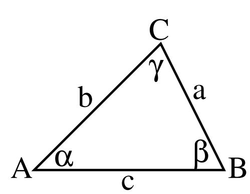
$$\sin 30^o = \frac{1}{2} \qquad \cos 30^o = \frac{\sqrt{3}}{2}$$

$$\sin 60^o = \frac{\sqrt{3}}{2} \qquad \cos 60^o = \frac{1}{2}$$

A handy little table to recall some important values of the main two trigonometrical functions

	0^o 30^o 45^o 60^o 90^o
sin	$\frac{\sqrt{0\ \ 1\ \ 2\ \ 3\ \ 4}}{2}$
cos	$\frac{\sqrt{4\ \ 3\ \ 2\ \ 1\ \ 0}}{2}$

3. Now consider the (not necessarily right) triangle $\triangle ABC$



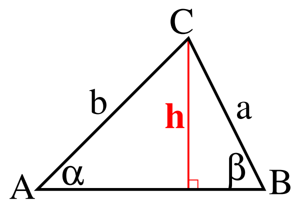
Theorem 6.6 (Law of sines). *In the previous picture the following holds*

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

Proof. We first prove

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$$

by dropping the altitude from C in the previous picture.



Then,

$$\sin\alpha = \frac{h}{b} \qquad \sin\beta = \frac{h}{a}$$

So, solving for h in these equations and setting them equal to each other we get

$$b\sin\alpha = a\sin\beta$$

which implies what we want.
The other part of the formula is proved similarly.

Theorem 6.7 (Law of cosines). *In $\triangle ABC$ (see picture above) the following holds*

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

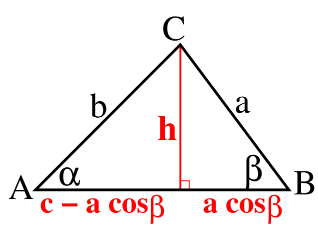
This law yields two more 'formulas' by rotating the triangle, they are

$$a^2 = b^2 + c^2 - 2bc\cos\alpha \qquad \textit{and} \qquad b^2 = a^2 + c^2 - 2ac\cos\beta$$

Proof. Clearly, it is enough to show

$$b^2 = a^2 + c^2 - 2ac\cos\beta$$

We look at the picture we used in the proof of the law of sines, and we notice that the base of the triangle has been cut into two pieces. We can find the lengths of these pieces by using the cosine of β, we get.



Using the Pythagorean theorem we get

$$a^2 = h^2 + (a\cos\beta)^2 \qquad b^2 = h^2 + (c - a\cos\beta)^2$$

Solving for h^2 in both equations and setting them equal to each other we get

$$a^2 - (a\cos\beta)^2 = b^2 - (c - a\cos\beta)^2$$

which implies

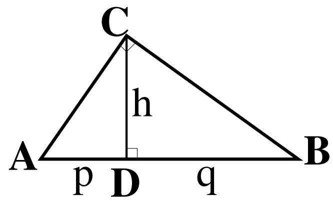
$$a^2 - (a\cos\beta)^2 = b^2 - c^2 + 2ac\cos\beta - (a\cos\beta)^2$$

Canceling $(a\cos\beta)^2$ we get

$$a^2 = b^2 - c^2 + 2ac\cos\beta$$

which implies the result we wanted.

4. **(Thales' theorem)** Let $\triangle ABC$ be the right triangle in the picture below



where h is the height, p is the length of $\overline{AD}$ and q is the length of $\overline{DB}$.
Then, since $\triangle ADC \sim \triangle CDB$ by AA, taking ratios of corresponding sides yields

$$\frac{p}{h} = \frac{h}{q} \qquad \text{or} \qquad h^2 = pq$$

Another cool property of right triangles is that, referring to the picture above, if M is the midpoint of $\overline{AB}$ then the segment $\overline{CM}$ is exactly half as long as $\overline{AB}$, thus after drawing this segment we have $\triangle ABC$ as the union of two isosceles triangles.

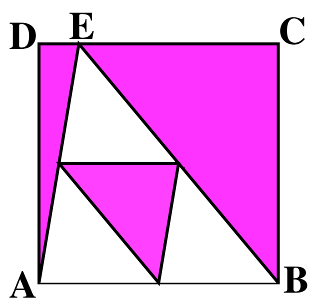
Problems

6.1. Let $\triangle ABC$ be an isosceles triangle with base $\overline{AB}$ and $\overline{AC} \cong \overline{BC}$. Prove that the triangle with base $\overline{AB}$ and sides given by the two angle bisectors of the base angles of $\triangle ABC$ is isosceles.

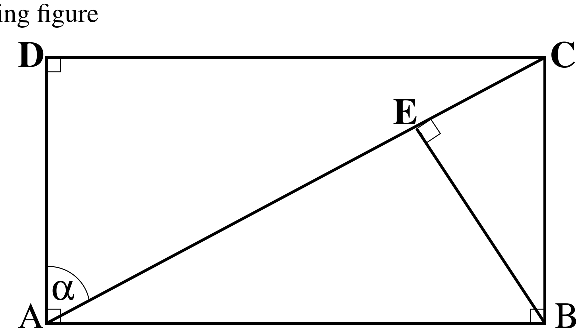
6.2. Oscar wants to find the height of a tree. He is exactly six feet tall, and notices that at the time he casts an eight-foot shadow. He measures the tree's shadow and finds that it is 74 feet long. How tall is the tree?

6.3. Agnes wants to tie a support line from the top of a 50 foot radio tower to an anchor spot 30 feet from the towers base. Approximately how long will the line need to be? (You might need a calculator for this one)

6.4. The triangle $\triangle ABE$ is inscribed within square $ABCD$ and has a height of 6 cm. Assuming that the little triangle inside $\triangle ABE$ is formed by midlines. What is the area of the shaded region?

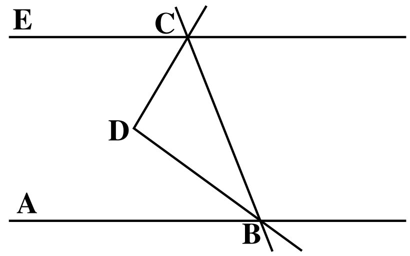


6.5. According to the following figure



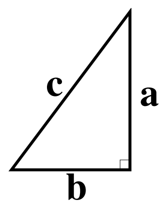
Assuming that $\alpha = 65°$. Give all pairs of congruent angles. Then give all pairs of similar triangles.

6.6. Assume that $\overline{AC}$ bisects $\angle ECB$, $\overline{EB}$ bisects $\angle ABC$, and that $\overline{AB}$ is parallel to $\overline{EC}$

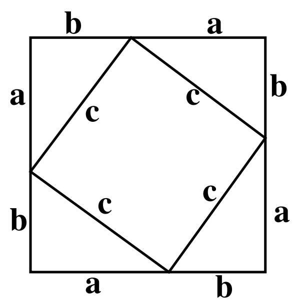


Show that $\triangle CDB$ is right.

6.7. Consider the following right triangle

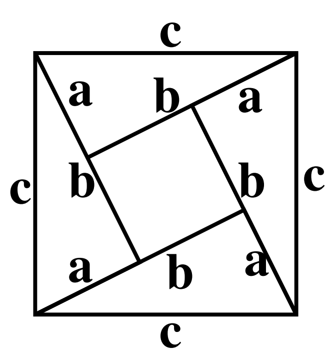


Now take four copies of it and arrange them in a square in the following way



First show that the square-like polygon that is contained in the large square is actually a square. Then compute the area of the big square in two ways. First by using the length of its side and then by computing the areas of the polygons (small square and triangles) that form it. Conclude the Pythagorean theorem.

Now with the same right triangle used above, use the following figure to find a different proof for the Pythagorean theorem.

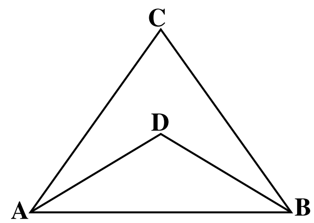


6.8. Does $Angle - Side - Side$ work as a criterion for congruence of triangles?

6.9. Is there a triangle with sides 2 *in*, 4 *in*, and 7 *in*?

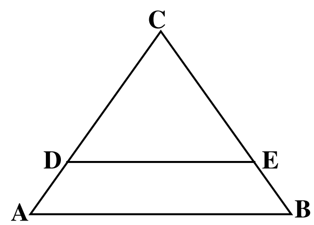
6.10. We know that if a right triangle has sides measuring a, b, and c then $a^2 + b^2 = c^2$. Show that if a triangle has sides measuring a, b, and c, and $a^2 + b^2 = c^2$ then the triangle must be right.

6.11. Assume that $\overline{AD}$ is the angle bisector of $\angle CAB$ and that $\overline{BD}$ is the angle bisector of $\angle ABC$.



Show that twice the measure of $\angle ADB$ equals the measure of $\angle ACB$ plus 180°.

6.12. Assume that $\overline{AB}$ is parallel to $\overline{DE}$

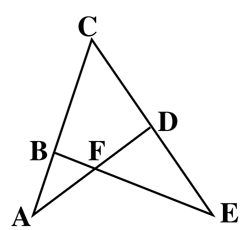


Show that if $\angle CDE$ and $\angle ABC$ are complementary then $\triangle ABC$ is right.

6.13. Let $ABCD$ be a rectangle with diagonals $\overline{AC}$ and $\overline{BD}$. Show that $\triangle ABC \cong \triangle DCB$.

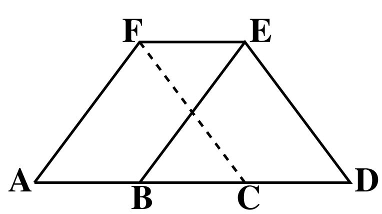
6.14. Show that any triangle $\triangle ABC$ can be broken into four congruent triangles, each one of them similar to $\triangle ABC$ in a 1 : 2 ratio.

6.15. In the figure below. Assume that $\overline{EB} \perp \overline{AC}$ and $\overline{AD} \perp \overline{CE}$ and $\overline{AB} \cong \overline{DE}$



Show that $\triangle CAD \cong \triangle CEB$.

6.16. I suspect the front and back of my tent are not congruent. So, I look at it against the sum and see



where any two segments that seem to be parallel are, in fact, parallel, and that $\overline{AB}$ is congruent to $\overline{CD}$.

Am I right assuming the back and front of my tent are not congruent?

Chapter 7
Quadrilaterals

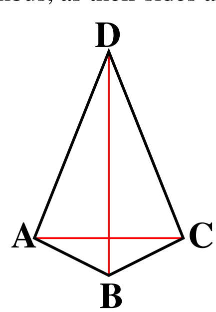
1. A polygon with 4 vertices (and sides) is called a **quadrilateral**.
2. Two sides of a quadrilateral that have a common vertex are called **adjacent sides** and two sides that do not have a common vertex are called **opposite sides**.
3. A **parallelogram** is a quadrilateral whose opposite sides are parallel.
4. Two angles that have their vertices at the endpoints of the same side of a parallelogram are called **consecutive angles**.
5. An **altitude** of a parallelogram is a segment that is perpendicular to two opposite sides (or the lines that contain them).
6. The area of a parallelogram with altitude measuring h and base b is $A = bh$.
   It is easy to see that a parallelogram with width larger than height has the area described above. If the parallelogram is a 'skinny' parallelogram, then the argument is a little more complex but it works anyway.
7. A **rectangle** is a parallelogram with all interior angles measuring $90°$.

   Remark 7.1. The diagonals of a rectangle are congruent. The converse is also true, i.e. if a parallelogram has congruent diagonals, then it must be a rectangle.

   Remark 7.2. It is in the definition that the interior angles of a rectangle are right, but it is also true that if a quadrilateral has four right angles then it must be a rectangle.

8. A **rhombus** is a quadrilateral with four congruent sides.

   Remark 7.3. The diagonals of a rhombus are perpendicular to each other. Moreover, they are angle bisectors. The converses of these two facts are also true but only when the quadrilateral is already a parallelogram. In fact, for a parallelogram to be a rhombus it is enough that **one** diagonal is a bisector.
   Now look at the figure below (a kite), in it we see that the diagonal $\overline{DB}$ is a bisector and that the two diagonals intersect in $90°$, however $ABCD$ is not a rhombus, as their sides are not congruent.



9. The area of a kite is given by

$$A = \frac{d_1 d_2}{2}$$

   where d_1 and d_2 are the lengths of the diagonals.

Proof. Consider the kite in the figure above, where d_1 is the length of the horizontal diagonal, and d_2 the length of the vertical one.
Since $\overline{AB} \cong \overline{BC}$, $\overline{AD} \cong \overline{DC}$, and $\overline{BD}$ is common to $\triangle BAD$ and $\triangle BCD$, then $\triangle BAD \cong \triangle BCD$. It follows that the area of the kite is twice the area of $\triangle BAD$, which is

$$\frac{d_2 \cdot \frac{d_1}{2}}{2} = \frac{d_1 d_2}{4}$$

The result follows. □

10. A **square** is a quadrilateral that is both a rhombus and a rectangle.

    Remark 7.4. If a rhombus has one right angle, then it is a square.

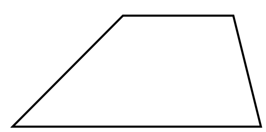
    Remark 7.5. Since a square is a rhombus, then we can compute its area using its diagonals.

11. A **trapezoid** is a quadrilateral with only two sides parallel. The two parallel sides of a trapezoid are called **bases** (sometimes I might call one the base and the other the **summit**).
12. The two angles adjacent to the base are called **base angles**, the two angles adjacent to the summit are called **summit angles**.

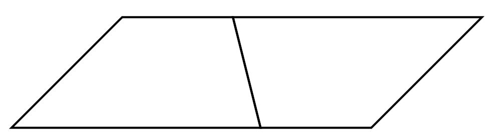
    Remark 7.6. Two consecutive angles in a trapezoid, not both base or summit angles, are suplementary.

    This is because the side of the trapezoid defines a transversal to the two bases, which are parallel.
13. An **altitude** of a trapezoid is a segment that is perpendicular to both bases (or to the lines containing them).
14. The area of a trapezoid is given by
$$A = \frac{h(b_1 + b_2)}{2}$$
    where h is the length of the altitude, and b_1 and b_2 represent the two bases.

    Proof. Consider the trapezoid



    We extend the base (summit) the length of the summit (base), then we join the end points to create the figure



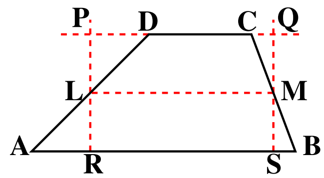
    Since the base and the summit are parallel, then it is easy to show that the two trapezoids in the picture (one upside down) are congruent and form a parallelogram. The are of this parallelogram is $(b_1 + b_2) \cdot h$, where b_1 and b_2 are the length of the base and summit of the trapezoid, and h is the height of the trapezoid. Since the parallelogram has twice the area of the trapezoid we are done. □

15. The **median** of a trapezoid is the segment joining the midpoints of the two non-parallel sides.

    Theorem 7.1. *The median is parallel to the bases and it measures exactly*

$$\frac{b_1 + b_2}{2}$$

    Proof. Consider A trapezoid $ABCD$ with median $\overline{LM}$. Draw two perpendicular lines to $\overline{AB}$ (and thus, also to $\overline{CD}$) from L and M. Finally, extend $\overline{CD}$ to a line. We obtain the following figure.



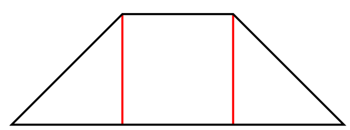
Since $\overline{AL} \cong \overline{LD}$ (L is the midpoint of $\overline{AD}$), $\angle ALR \cong \angle PLD$ (vertical angles), and both $\angle DPL$ and $\angle LRA$ are right, then $\triangle ALR \cong \triangle DLP$ by SAS. Similarly, $\triangle CQM \cong \triangle BSM$.
It follows that $\overline{PL} \cong \overline{LR}$ and $\overline{QM} \cong \overline{MS}$, and thus $\overline{LM}$ is a median of rectangle $RSQP$. Hence, $\overline{LM}$ is parallel to the summit and base of the rectangle (which are the same lines giving us the base and summit of the trapezoid). We also get that $\overline{LM} \cong \overline{PQ} \cong \overline{RS}$, and thus $\overline{LM}$ is one-half of the sum of the lengths of $\overline{PQ}$ and $\overline{RS}$. But, since $\triangle ALR \cong \triangle DLP$ and $\triangle CQM \cong \triangle BSM$ then the sum of the lengths of $\overline{PQ}$ and $\overline{RS}$ is equal to the sum of the lengths of $\overline{DC}$ and $\overline{AB}$ □

Remark 7.7. (i) If m is the length of the median of a trapezoid with height h, then we can say that the area of a trapezoid is $A = mh$.
(ii) Another proof of theorem 7.1 may be obtained by using the techniques in chapter 10.

16. An **isosceles trapezoid** is a trapezoid whose non-parallel sides are congruent.

Remark 7.8. In an isosceles trapezoid the two base angles are congruent, and so are the two summit angles.

This follows from the picture below



where the two red lines are heights. Since the trapezoid is isosceles, then the two triangles are congruent (they are right triangles with two congruent corresponding sides). So, the base angles of the trapezoid are congruent.

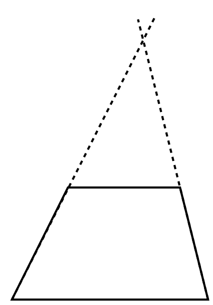
Remark 7.9. The diagonals of an isosceles trapezoid are congruent.

If you draw the diagonals you will obtain two congruent triangles, formed by the base of the trapezoid, a diagonal and a side. Hence, the diagonals are congruent.

Remark 7.10. Any two opposite angles in an isosceles trapezoid are suplementary.

This follows from the fact that the two summit angles are congruent and the two base angles are congruent, and remark 7.6.

17. In case you are like me and forget so many properties about trapezoids, just keep in mind the following picture.



which shows how a trapezoid can be 'extended' to a triangle, and since the bases of the trapezoid are parallel, then the big triangle is similar to the small (dashed) one... pretty much all you need to know can be obtained by from this picture.

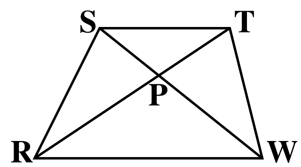
Problems

7.1. Oscar wants to cover a football field with sod. The field is 360 feet long and 200 feet wide. Sod can be purchased in squares in 1 foot increments from 2 feet wide up to 9 feet wide. What is the largest size squares Oscar can purchase with which he can cover the field completely without any gaps or overhang?

7.2. What is the perimeter of a rhombus with diagonals measuring 32 *in* and 24 *in*.

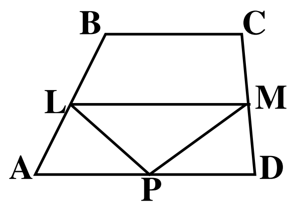
7.3. Find the height of an isosceles trapezoid with base 22 *in*, summit 8 *in* and other two sides measuring 25 *in*.

7.4. The trapezoid $RWTS$ is isosceles



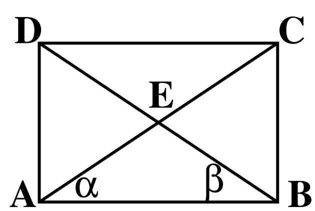
Is $\triangle RPW$ isosceles?

7.5. Let $\overline{LM}$ be the median of the trapezoid $ABCD$, and P the midpoint of $\overline{AD}$

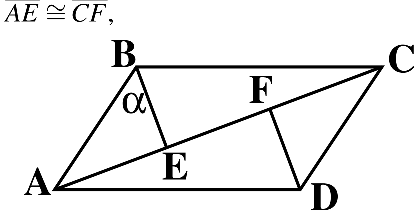


If $\overline{LP} \cong \overline{MP}$. Show that $ABCD$ is isosceles.

7.6. If $\alpha > \beta$. Can $ABCD$ be a rectangle?

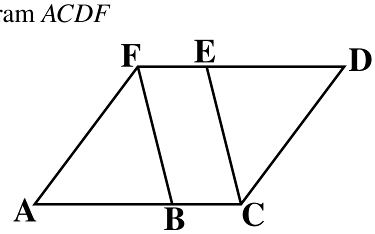


7.7. Let $\#ABCD$ be a parallelogram, $\overline{AE} \cong \overline{CF}$,



(a) Find $\angle CDF$ in terms of α.
(b) Show $\angle ABE \cong \angle CDF$.

7.8. Consider the following parallelogram $ACDF$



(a) Assume that $\overline{AB} \cong \overline{ED}$. Show that $BCEF$ is a parallelogram.
(b) Assume that $\overline{FB}$ and $\overline{EC}$ bisect $\angle AFD$ and $\angle ACD$ respectively. Show that $BCEF$ is a parallelogram.

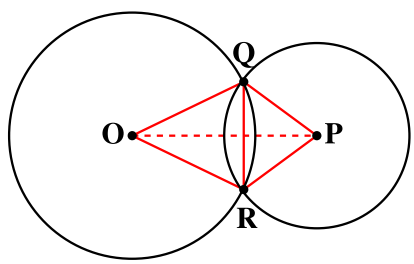
Chapter 8
Circles

8.1 Basic properties of circles

1. A circle is the set of all points that are at the same distance r from a given point C. The number r is called the radius of the circle and C is the center. The largest distance between two points on a circle is $d = 2r$, which is called the diameter of the circle.

   Remark 8.1. Two distinct circles intersect in at most two points.

   Proof. We have examples of two circles intersecting in zero, one or two points. Assume two circles intersect in at least two points. We get the picture

   where the dashed line (joining the centers of the circles) is the altitude for both isosceles triangles $\triangle ROQ$ and $\triangle RPQ$. It follows that once the point R (for example) has been determined, then there is a unique other point of intersection... given by the dashed line and R. It follows that two circles can intersect in at most two points. □

2. A line intersects a circle C in at most two points. A line that does not intersect C is said to be exterior, a line that intersects C in exactly one point is a tangent, and a line intersecting C is called a secant.
3. The segment, created by a secant line, that is between two points of the circle is called a chord.

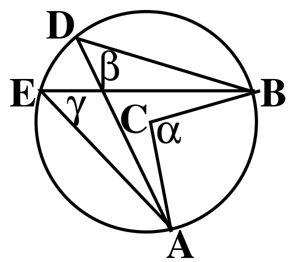
   Remark 8.2. If a chord is perpendicular to a radius, then the radius bisects the chord

   Proof. Joining the center with the extremes of the chord we will create a triangle that has altitude the radius that is perpendicular to the chord. Since this triangle is also isosceles, it follows that the altitude must bisect the base (which is the chord). □

   Remark 8.3. If two chords are at the same distance from the center then they are congruent.

   Proof. Recall that distances from a point to a segment/line are measured using segments that are perpendicular to the original line. In this case the radius is that perpendicular segment. Using the previous problem we get that the distance to the center is actually the height of the isosceles triangle created by joining the center with the extremes of a chord. Since the sides of these triangles are uniquely determined (they are radii) and the height is also unique, then their bases must be uniquely determined. Done. □

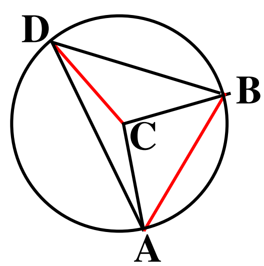
4. Let us recall that there is a second way to measure angles, by using radians. We use radians when we want to use length (inches, yards, centimeters, etc) to measure angles. The conversion rate from degrees to radians is given by $180^\circ = \pi$ *radians*.
5. An angle with vertex at the center of the circle is called a central angle. An angle with vertex at a point of the circle is called an inscribed angle. In the picture below $\angle ACB$ is central, and both $\angle AEB$ and $\angle ADB$ are inscribed.



Remark 8.4. Note that the angles in the picture above share points A and B, when this happens we get that the central angle is exactly twice the inscribed angle. That is,

$$2\gamma = 2\beta = \alpha$$

Proof. Just consider one inscribed angle and a central angle. Draw a radius from the center to the vertex of the inscribed angle, and $\overline{AB}$ creating three isosceles triangles.



From the picture we get (and using $\triangle DAC$ and $\triangle BDC$ are isosceles)

$$\begin{aligned} 2\angle ADC + \angle DCA &= 180^\circ \\ 2\angle BDC + \angle DCB &= 180^\circ \end{aligned}$$

which implies

$$2\angle ADC + 2\angle BDC + \angle DCA + \angle DCB = 360^\circ$$

Also, the three angles around the center add up to 360°, so

$$\angle DCB + \angle DCA = 360^\circ - \angle ACB$$

Plugging this into the previous equation we get

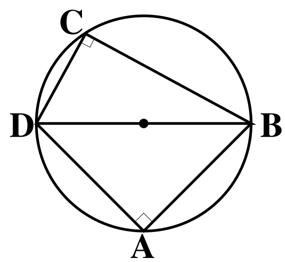
$$2\angle ADC + 2\angle BDC + (360^\circ - \angle ACB) = 360^\circ$$

and this implies

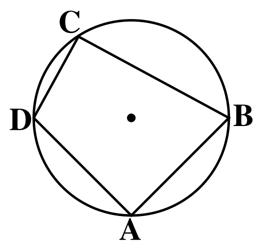
$$2(\angle ADC + \angle BDC) = \angle ACB$$

which is exactly what we wanted to prove. □

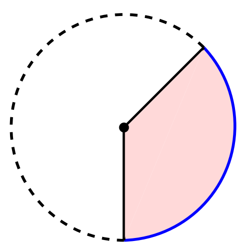
Remark 8.5. Using the previous remark we can see that any triangle inscribed in a circle with the diameter as a base must be a right angle. That is, in the picture below, $\overline{DB}$ is the diameter and both triangles $\triangle DBC$ and $\triangle DBA$ are right.



Remark 8.6. Any pair of opposite angles in a quadrilateral inscribed in a circle are supplementary. That is, in the picture below $\angle D + \angle B = \angle C + \angle A = 180°$.



6. The points on a circle that lie between the two points a central angle intersects the circle is called a circle arc. The whole 'slice' obtained by drawing a central angle is called a circle section. So, in the next picture, the shaded area is the slice given by the central angle and the blue (continuous line) is the arc determined by the central angle.



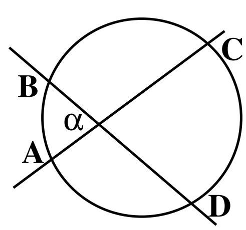
7. The perimeter (or circumference) of a circle with radius r is $P = 2\pi r$. The area of a circle is $A = \pi r^2$.

   Remark 8.7. The length of a circular arc with central angle α **(in radians)** is $\ell_\alpha = \alpha r$. The area of a circle sector with central angles α is $A_\alpha = \dfrac{\alpha r^2}{2}$.
   It follows that congruent central angles yield congruent arcs and sectors.

   This is just a simple application of proportions.

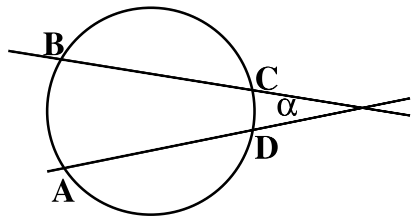
8.2 Tangents, secants and chords of circles

1. In the picture below, α equals one-half of the sum of the central angles that are determined by the arcs AB and CD.



Proof. Let P be the intersection of $\overline{BD}$ and $\overline{AC}$, and join A with D creating $\triangle APD$. Note that $\angle PAD$ measures half the length of arc CD, and that $\angle PDA$ measures half the length of arc AB. Since α is exterior to $\triangle APD$ with opposite interior angles $\angle PAD$ and $\angle PDA$. The results follows. □

2. In the picture below, α equals one-half of the difference of the central angles that are determined by the arcs AB and CD.



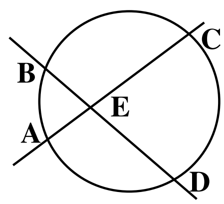
Proof. Let P be the vertex of the angle α. Join A with C creating $\triangle APC$. Note that $\angle ACB$ is external to $\triangle APC$. It follows that $\alpha + \angle CAP = \angle ACB$, or

$$\alpha = \angle ACB - \angle CAP$$

The result follows from the fact that arc AB is held by $\angle ACB$, and arc CD is held by $\angle CAP$. □

3. In the picture below,

$$\overline{BE} \cdot \overline{ED} = \overline{AE} \cdot \overline{EC}$$



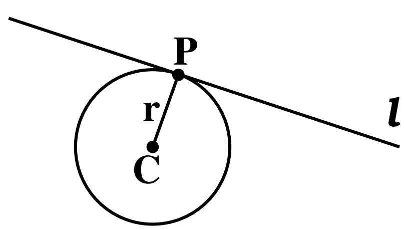
(note that I am talking about the multiplication of the **lengths** of those segments)

Proof. Since lengths of segments are in the result, then we suspect we need to get a couple of similar triangles going on here.
Join A with B and C with D creating $\angle ABD$ and $\angle ACD$. Since these angles hold the same arc (AD), then they are congruent. Moreover, $\angle BEA \cong \angle CED$, thus $\triangle BEA \sim \triangle CED$. Looking at the ratios of their corresponding sides we get

$$\frac{BE}{CE} = \frac{AE}{ED}$$

which implies what we want. □

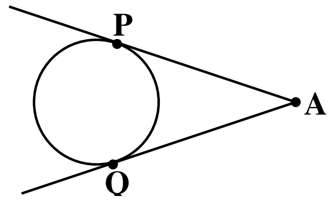
4. A tangent line to a circle at a point P is perpendicular to the radius that contains P. That is, in the picture below, where l is tangent to the circle with center C at the point P,



the line l is perpendicular to $\overline{CP}$

Proof. If this were not true, then we draw a perpendicular to l falling from C. Call Q to the point of intersection of these two perpendicular lines. It follows that the (shortest) distance from C to l is the length of $\overline{CQ}$. Thus, the length of $\overline{CP}$ (which is r) must be larger than the length of $\overline{CQ}$. It follows that Q must be inside the circle, which will force the line to be a secant. This is not possible. Hence $\overline{CP}$ must be perpendicular to l. □

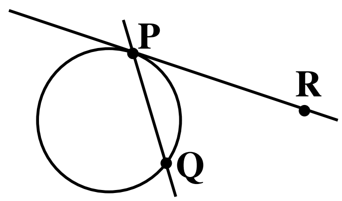
5. Let us throw two tangent lines from a point A outside the circle



Then $\overline{AP} \cong \overline{AQ}$.

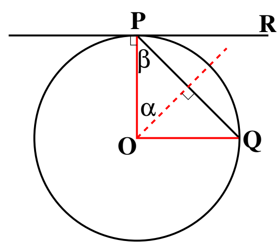
Proof. Let O is the center of the circle. Note that $\triangle OPA$ and $\triangle OQA$ are right triangles with a common hypothenuse. Moreover, $\overline{OP} \cong \overline{OQ}$ because both are radii of the circle. It follows from the Pythagorean theorem that $\overline{AP} \cong \overline{AQ}$. □

6. In the following picture, where $\overleftrightarrow{PR}$ is tangent to the circle at P,



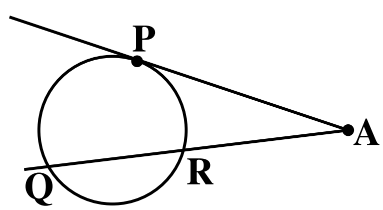
the angle $\angle QPR$ is one-half the central angle with arc PQ.

Proof. Let O be the center of the circle. If P, O and Q are collinear then the result is clear.
Assume P, and O and Q are not collinear. Draw radii $\overline{OP}$ and $\overline{OQ}$, creating an isosceles triangle $\triangle POQ$. The altitude of this triangle must intersect $\overline{AP}$ at a point different from P because P, and O and Q are not collinear. In fact the altitude must bisect $\overline{PQ}$. We get this picture.



It follows that $\angle PTR$ is right and thus $\angle TPS = \alpha$. We are done because $\angle POQ = 2\alpha$. □

7. In the following picture

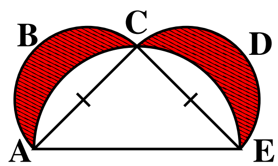


$$\frac{\overline{AQ}}{\overline{AP}} = \frac{\overline{AP}}{\overline{AR}}$$

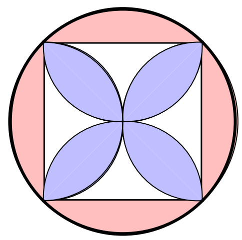
Proof. Just as like in a previous problem, we need two similar triangles. Looking at the final formula we can guess what triangles we need to work with. We will show $\triangle QPA \sim \triangle PRA$.
We draw segments $\overline{PR}$ and $\overline{PQ}$ creating a couple of triangles. Using the previous problem we get that $\angle PQR \cong \angle RPA$.
Since $\angle PAQ$ is common to both triangles, then by AA we get $\triangle QPA \sim \triangle PRA$. Done. □

Problems

8.1. Assume that ABC, ACE, and CDE are semicircles, and that $AC \cong CE$. Show that the area of the shaded region is equal to the area of $\triangle ACE$



8.2. Consider the following figure

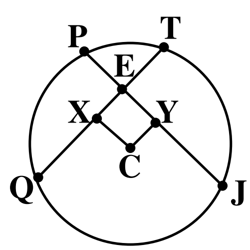


which is obtained by drawing a square inscribed in a circle and then four semicircles with diameters the sides of the square.

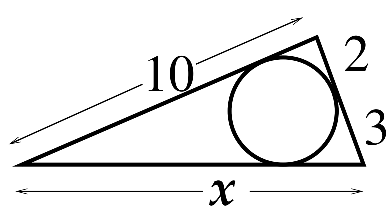
Show that the red area is equal to the blue area.

8.3. Show that if two chords of the same circle are congruent then the arcs they determine must be congruent as well.

8.4. Assume that $CXEY$ is a square and that C is the center of the circle. Show that the arc QP is congruent to the arc JT.



8.5. Find x



8.6. From a point 2 units from a circle, a tangent is drawn. If the radius of the circle is 8 units, find the length of the tangent segment.

8.7. Find the circumference of a circle that is circumscribed about a square whose perimeter is 36 *in*.

8.8. Let p and q be two positive numbers. Prove that

$$\sqrt{pq} \leq \frac{p+q}{2}$$

Hint: Use Thales' theorem (chapter 6) and remark 8.5.

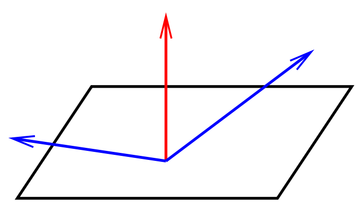
Chapter 9
3-D geometry

9.1 Planes and lines

1. Let us called $\mathbb{E}^3$ to our three-dimensional space. $\mathbb{E}^3$ is formed by points, lines, planes and many other curves and solids.
2. Two distinct lines in $\mathbb{E}^3$ can be skew (no intersection and pointing in different directions), parallel (no intersection and pointing in the same direction), or they intersect in a point.
3. Just as an infinite number of points create a line, an infinite number of lines create a plane, which only has two dimensions (width and length).

   Remark 9.1. If two distinct planes in $\mathbb{E}^3$ intersect, then they do so along a line. Also, two distinct planes could be parallel (meaning no intersection).

4. The normal vector (or line) of a plane is a vector (line) that is perpendicular to all lines on the plane. For instance, in the picture below, the normal vector would be the red one (the middle one) and not the other two (blue ones)... they just don't seem to be perpendicular enough!



5. The dihedral angle between two planes is the angle formed by their normal vectors.

9.2 Solids

1. A prism is a solid with parallel congruent bases (most of the times polygons)

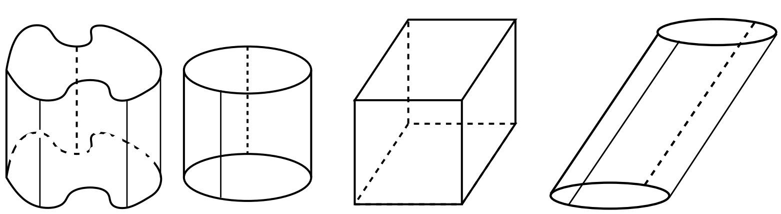


Fig. 9.1 Prisms

A prism with bases that are not aligned one directly above the other is called an oblique prism.

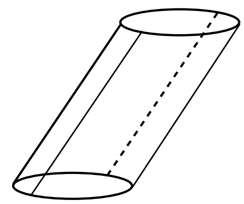


Fig. 9.2 Oblique prism

Remark 9.2. A prism with a polygonal base has parallelograms as lateral faces.

Examples of prisms are: cube, rectangular solid (shoe box), cylinder, etc.

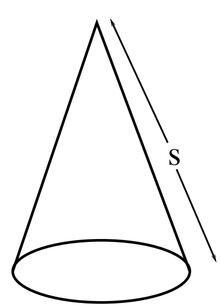
2. The surface area of a solid is the sum of the areas of all its sides. The volume of a solid measures the amount of space it occupies in space.

There are many formulas to find the surface area and volume of the most common solids, here it is a list.

	Prism	*Cone*	*Pyramid*	*Sphere*
Volume	$h \cdot A_{base}$	$\frac{1}{3}h\pi r^2$	$\frac{1}{3}h \cdot A_{base}$	$\frac{4}{3}\pi R^3$
Surface area	$h \cdot P_{base} + 2A_{base}$	$\pi rs + \pi r^2$	$\star$ (see below)	$4\pi R^2$

where

a. A_{base} is the area of the base of the solid,
b. P_{base} is the perimeter of the base of the solid,
c. h is the height of the solid,
d. r is the radius of the circle that is the base of the cone,
e. R is the radius of the sphere
f. s is the length of the slant height of a cone (see picture below)

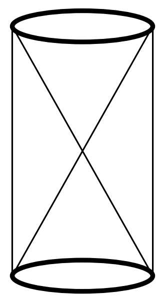


$\star$ The surface area of the pyramid depends on the number of sides, and the shape, of the polygon used for the base.

Problems

9.1. Two opposite vertices in a rectangular box are at distance $7\sqrt{2}$ *in*. If the sides of the box are in the ratio 1 : 2 : 3, then what is the volume of the box?

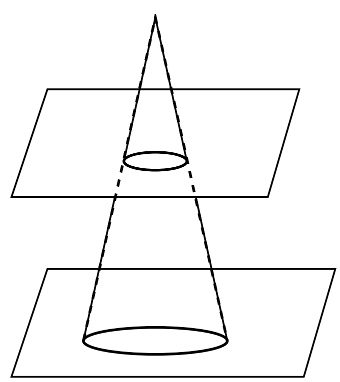
9.2. Consider a 'sand-clock-like' figure formed by two (not necessarily) congruent cones sharing their vertex. This figure is perfectly fit inside a cylinder that has the same base as the cones'... something like the following picture.



What portion of the volume of cylinder is the volume of the sand clock?

9.3. A food company wants to paint their entire tomato paste can red, including the top and bottom. If the can is a cylinder with lid's diameter equal to 4 *cm*, and height 7 *cm*, what is the total surface area that will need to be painted?

9.4. Consider a cone that has been cut/sliced by a plane that is parallel to its base, like in the picture below



If the height of the cone is 12, the height of the little cone that is formed above the highest plane is 4, and the slant height of the big cone is 15, then what is the ratio of the surface areas of the cones?

9.5. If the edge of a cube is y *in*. What is the distance from one corner of the cube to the furthest corner on the opposite side of the cube?

9.6. The radius of a sphere is tripled, by what number is its volume multiplied?

9.7. If a cube and a $3 \times 8 \times 9$ inches rectangular box have the same volume. What is the area of a side of the cube?

9.8. The volume of a cylinder is 5076 ft^3. Its base radius and height are in a ratio of 3 : 10. Find the surface area of the cylinder.

9.9. A bowl in the shape of a half-sphere with radius 5 *in* is full of water. If the radius of the base of a cylindrical container is 2 *in*, how tall does it need to be to contain all the water in the bowl?

Chapter 10
The Cartesian plane

In this section we will learn enough to use coordinates to prove geometric properties of triangles, quadrilaterals, and many other shapes. We will learn more techniques in coordinate geometry in the section on conics.

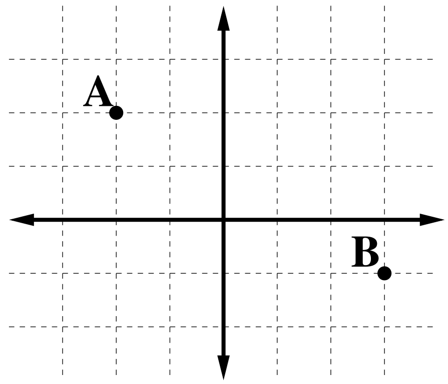
1. We can write every point P on the plane as an ordered pair of numbers $P = (x,y)$. The numbers x and y are called the coordinates of P.
   The converse is also true, given a pair (x,y) then it is possible to graph the pair as a point on the plane.
   This presentation of the plane as a set of ordered pairs is called the Cartesian plane.
2. Given two points $P = (x_1,y_1)$ and $Q = (x_2,y_2)$ then we can find the distance between P and Q by using the formula
   $$d(P,Q) = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$
3. The midpoint of the segment $\overline{PQ}$, where $P = (x_1,y_1)$ and $Q = (x_2,y_2)$ is the point
   $$M = \left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right)$$
4. The equation of a line is given by $y = mx+b$, where m is the slope of the line and b gives the y-intercept of the line.
   The slope of the line through the points $P = (x_1,y_1)$ and $Q = (x_2,y_2)$ is
   $$m = \frac{y_2-y_1}{x_2-x_1}$$
   Remark 10.1. The slope determines the angle the line makes with the x-axis. In fact $m = \tan\alpha$, where α is the angle formed by a line with slope m and the x-axis.

   Remark 10.2. Two lines that are parallel have the same slope.
   The product of the slopes of two lines that are perpendicular is equal to -1.

With the tools learned above we can show things such as: (1) the midline of a trapezoid is parallel to the base and its length is equal to one-half of the sum of the bases, (2) if the diagonals of a rectangle are perpendicular, then the rectangle is a square, (3) the line segments joining the midpoints of opposite sides of any quadrilateral bisect each other, (4) the diagonals of a rectangle are equal in length, (5) the diagonals of a rhombus are perpendicular, etc.

Problems

10.1. What is the slope of the line joining A and B?



What is the equation of the line? Give the equation of the line perpendicular to the line through A and B that passes through $(1,2)$. Find the intersection point of the two lines.

10.2. Use coordinates to prove that the altitudes of a triangle meet at one point.

10.3. Show that the midline of a trapezoid is parallel to the base and its length equals one-half of the sum of the lengths of the bases.

10.4. Show that if the diagonals of a rectangle are perpendicular, then the rectangle is a square.

10.5. Show that the line segments joining the midpoints of opposite sides of any quadrilateral bisect each other.

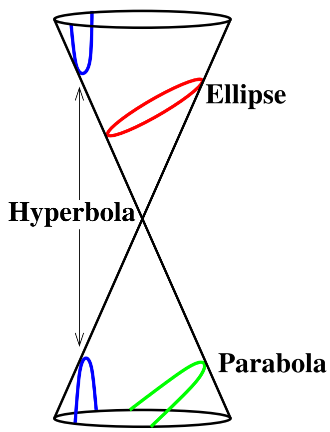
10.6. Show that the diagonals of a rectangle bisect each other.

10.7. Show that the diagonals of a rhombus are perpendicular.

10.8. Let $A = (a, a+1)$ and $B = (3a+5, a-1)$.
(a) Find the equation of the line through A and B.
(b) Find the equation of the line perpendicular to $\overleftrightarrow{AB}$ and through A.
(c) Find the equation of the line perpendicular to $\overleftrightarrow{AB}$ and through B.

Chapter 11
Conic sections

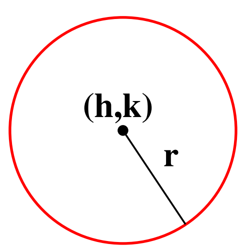
Conic sections are the curves formed when a plane intersects a double cone like in the picture below. The intersections are of three types; ellipses, hyperbolas or parabolas



Note that the base (or summit) of the double cone is a circle, so we could consider a circle to be a fourth type of conic section, but a circle is just a particular type of ellipse.

Now we know why these curves are called conic sections. However, this is not the way we usually think about them. We want to study these curves in a coordinate plane so we can find their equations, just as we did previously with lines.

Definition 11.1. A circle is a set of points of the Cartesian plane that are at the same distance r from a point C. The number r is called the radius of the circle and C is called the center of the circle.



Circle with radius r and center (h,k).

Since every point (x,y) on the circle above is at distance r from (h,k), then

$$\sqrt{(x-h)^2+(y-k)^2} = r$$

which by squaring both side yields what is known as the equation of that circle.

$$(x-h)^2+(y-k)^2=r^2$$

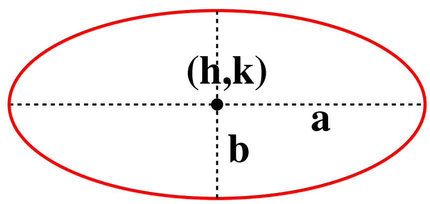
Remark 11.1. Note that after foiling the equation of a circle we get that the terms with x^2 and y^2 have the same coefficient (one) and same sign (positive).

Definition 11.2. An ellipse consists of all points such that the sum of the distances to two fixed points, F_1 and F_2 is constant. The points F_1 and F_2 are called the foci (the singular is focus) of the ellipse.

Before getting the equation of an ellipse we need to set a little notation:

1. The center of the ellipse is the midpoint of $\overline{F_1F_2}$. The distance from a focus to the center is denoted c.
2. The points where the line $\overleftrightarrow{F_1F_2}$ (called the major axis) intersects the ellipse are also equidistant from the center. The distance from one of these points to the center is denoted a.
3. The perpendicular line to $\overleftrightarrow{F_1F_2}$ (called the minor axis) at the center of the ellipse intersects the ellipse in two points that are also equidistant from the center. The distance from one of these points to the center is denoted b.

The following picture shows a ‘horizontal’ ellipse with center (h,k) and distances a and b on the axes



Note that if the ellipse were ‘vertical’ then we should interchange the letters a and b, as a should always be the larger of the two distances. These changes are mostly cosmetic, and thus we will focus on finding the equation of a ‘horizontal’ ellipse just like the one in the picture above.

We first notice that if we found the equation of an ellipse centered at $(0,0)$ then we could just translate this ellipse, and its equation, to another congruent ellipse with center at any other point (see chapter 12).

As an ellipse is the set of points with same sum of distances to the foci, then we consider the two points at distance a from the center and see how far it is from the foci. It is easy to see that the distance to one of them is $a-c$ and to the farthest is $a+c$. Hence, the sum of the distances to the foci of all points in an ellipse is always $2a$. If we now look at the two points on the ellipse at distance b from the center we can see that it is equidistant to the foci. Moreover, the segment joining a focus and this point (this segment has length a) is the hypothenuse of a right triangle with legs b and c. It follows, using the Pythagorean theorem, that

$$b^2+c^2=a^2$$

This equality is needed to get the equation of the ellipse. In order to find this equation we use the definition of an ellipse, and we use that the foci are at distance c from $(0,0)$ (the center of the ellipse) and the ellipse is ‘horizontal’ to get that the foci are $(\pm c,0)$. It follows that any point on the ellipse satisfies

$$d((x,y),(-c,0))+d((x,y),(c,0))=2a$$

or, in other words

$$\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a$$

After quite a bit of algebra (see example 11.1) we obtain

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

which, when translated to an ellipse centered at (h,k) yields

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

which is what it is known as the equation of the ellipse.

Don't forget that all the work above is for a 'horizontal' ellipse. See problem 11.6.

Example 11.1. Let us find the equation of the ellipse with foci $(1,1)$ and $(1,5)$ and sum of distances to the foci equal to 10

We set the distance equation we discussed above.

$$d((x,y),(1,1))+d((x,y),(1,5))=10$$

which implies

$$\sqrt{(x-1)^2+(y-1)^2}+\sqrt{(x-1)^2+(y-5)^2}=10$$

we move one of the radicals to the right hand side to get

$$\sqrt{(x-1)^2+(y-1)^2}=10-\sqrt{(x-1)^2+(y-5)^2}$$

Now we square both sides and simplify

$$(x-1)^2+(y-1)^2=10^2-20\sqrt{(x-1)^2+(y-5)^2}+(x-1)^2+(y-5)^2$$

$$(y-1)^2-(y-5)^2-100=-20\sqrt{(x-1)^2+(y-5)^2}$$

$$8y-124=-20\sqrt{(x-1)^2+(y-5)^2}$$

we simplify a little more and we square

$$(2y-31)^2=25((x-1)^2+(y-5)^2)$$

Now we just have to clean this up a little, completing the squares on the way,

$$25(x^2-2x+4)+21(y^2-6y+9)=600$$

Hence,

$$\frac{(x-2)^2}{24}+\frac{(y-3)^2}{200/7}=1$$

Note that, in this case, the center is $(2,3)$, $a=\sqrt{24}$ and $b=\sqrt{200/7}$.

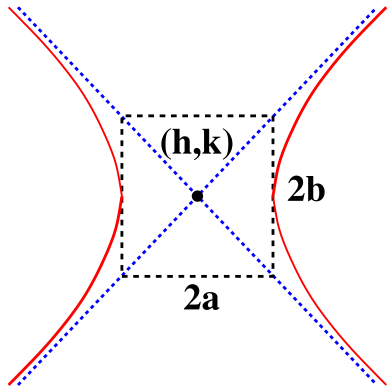
Remark 11.2. Note that after foiling the equation of an ellipse we get that the terms with x^2 and y^2 have distinct coefficients (if they were the same then we get a circle) but with the same sign (positive).

Definition 11.3. A hyperbola is the set of points such that the difference of the distances to two fixed points, called F_1 and F_2, is constant. These fixed points are called the foci of the hyperbola.

Before getting the equation of a hyperbola we need to set a little notation:

1. The center of the hyperbola is the midpoint of $\overline{F_1F_2}$. The distance from a focus to the center is denoted c.
2. The points where the line $\overleftrightarrow{F_1F_2}$ intersects the hyperbola, called the vertices of the hyperbola, are also equidistant from the center. The distance from one of these points to the center is denoted a.
   It is easy to see that the difference of the distances from a vertex to the foci is going to have to be $2a$. Hence, the difference of distances to the foci that is the same for all points on the hyperbola must be $2a$.
3. There are two lines (called the asymptotes of the hyperbola) that set the limits for the graph of the hyperbola. The hyperbola will get as close as one wants from these lines but will never touch or cross them.
4. We could draw a rectangle with vertices on the asymptotes and that is tangent to the hyperbola at the vertices. This rectangle has base $2a$ and height $2b$. This number b will be important later.

The following picture shows a ‘horizontal’ hyperbola (in red) with center (h,k), distance to the vertices a. It also features the asymptotes (in blue), and the rectangle that shows how to find $2b$



Note that this picture gives us enough information to find the equations of the asymptotes. In particular the slopes of these lines are $\pm\frac{b}{a}$.

By doing the same type of argument used to find the equation of the ellipse we can find the equation of a ‘horizontal’ hyperbola centered at (h,k) and distances a, and b as described above:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Moreover, it is possible to show that $a^2 + b^2 = c^2$.

It is important to mention that for all this I am considering the hyperbola to open East and West. If you are working with a hyperbola that opens North and South, then you should look at it sidewise to figure where the foci will be.

Remark 11.3. Note that after foiling the equation of the hyperbola we get that the terms with x^2 and y^2 have different signs.

Definition 11.4. A parabola is the set of points that are equidistant from a given point F and a given line ℓ. The point is called the focus of the parabola, and the line is called its directrix.

Before getting the equation of a parabola we need to set a little notation:

1. Let P be the point on ℓ such that $d(P,F) = d(P,\ell)$. The midpoint of $\overline{PF}$ is called V, and it is the vertex of the parabola.
2. The distance from the vertex to the directrix is denoted p. Clearly the distance between F and V is also p.
3. The line $\overleftrightarrow{VF}$ is called the axis of the parabola. It is perpendicular to the directrix.

In this book we will only consider horizontal or vertical axis and, just like we have done before, we will only find the equation of one of these cases, and the other equation will be left as an exercise.

Consider a parabola with vertex $V = (h,k)$ and horizontal directrix (and thus a a vertical axis) and distance p as above. The equation of this curve is

$$4p(y-k) = (x-h)^2$$

Note that the distance p might need to be considered to be negative in case the parabola opens downwards instead of upwards.

Remark 11.4. The general equation of a conic section (of the types we are considering) will have the form

$$Ax^2 + By^2 + Dx + Ey + F = 0$$

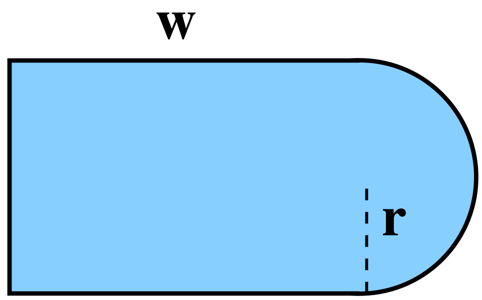
Note that just by looking at the coefficients with x^2 and y^2 and their signs we can tell what type of conic the equation will yield.

Problems

11.1. Given the equation $4x^2 + 9y^2 = 36$. Find the standard form of this ellipse's equation. Then find the coordinates of the foci and sketch the graph of the ellipse.

11.2. Repeat the previous problem with $4y^2 + 9x^2 = 36$.

11.3. Consider a pool in the shape of a rectangle with a semicircle on the side, as in the picture below.



Find the maximum area for a pool that must have perimeter equal to 18 meters.

11.4. Find the foci of the ellipse $\dfrac{(x-1)^2}{5} + \dfrac{(y+1)^2}{3} = 1$.

11.5. Find the equation of a circle having a diameter with endpoints $(1,1)$ and $(5,4)$.

11.6. Find the equation of a 'vertical' ellipse. Find the relation between a, b and c. Repeat the problem with a 'vertical' hyperbola.

11.7. Find the equation of a parabola with horizontal axis.

11.8. If the coordinates of the extremes of the diameter of a circle are $(3,1)$ and $(6,5)$, what is the equation of the circle?

11.9. Find the equation of **an** ellipse that passes through the points $(-1,2), (5,2), (2,3)$, and $(2,1)$.

Chapter 12
Transformations

In this chapter we will learn about movements of the plane and what they do to points and lines living on it. These movements will be called transformations, and they will be though of as functions. Specifically, a transformation of the plane is a bijective function from the plane onto the plane.

Definition 12.1. An isometry of the plane is a transformation of the plane that moves everything on the plane simultaneously and that preserves distances.

Equivalently, an isometry is a bijective function $\phi : \mathbb{E}^2 \to \mathbb{E}^2$ such that

$$d(x,y) = d(\phi(x), \phi(y))$$

Example 12.1. For example, the shifting of the plane one unit to the right is an isometry. Also, a rotation in 30° around the origin is also an isometry.

Note that if we first rotate in 30° and then we shift we get another isometry. Similarly, if we first shift, then rotate and then shift again, and then we rotate twice, then we get an isometry.

Remark 12.1. A transformation that preserves distances will also preserve angles.

We can re-define congruence of objects as $A \cong B$ if there is an isometry mapping A to B bijectively. Thus, the transformation does the 'we move A over B' part in our intuitive definition of congruency.

Definition 12.2. The basic isometries of the plane are

1. The shifting of the plane in any (straight line) direction is called a translation. If a shifting sends the origin to a point P, then we denote this translation as T_P.
2. A (counterclockwise) rotation in an angle θ around a point P will be called $R_{\theta,P}$, or just R_θ when it is clear what the point P is.
3. A reflection is the transformation that consist in using a line as a mirror to reflect what is in one side of it to its other side, and vice-versa. We will say that the reflection is across the line ℓ, and we will denote this as J_ℓ.

Remark 12.2. A translation fixes no points, i.e it moves all the points of the plane, a rotation fixes exactly one point (the point P), and a reflection fixes the points of the line ℓ and nothing else.

As mentioned in example 12.1, isometries can be used in succession. Since isometries are functions, then using an isometry after the other means that the corresponding functions have been composed. In fact, if ϕ and ψ are two isometries of the types listed above then performing ϕ followed by ψ always yields another isometry (that might not be of the type described above). For example if we follow a reflection across the x-axis by a translation that is parallel to the x-axis we obtain an isometry that is not a reflection, translation or rotation. This type of isometry is called a glide reflections.

Also, a rotation with center not equal to the origin can be written as a composition of translations and a rotation with center the origin. Similarly, a reflection can be written as a composition of translations, rotations and a reflection across the x-axis (or the y-axis if you please).

Remark 12.3. Every isometry of the plane can be constructed by a succession of rotations, translations, and reflections.

In fact, reflections are enough, as two reflections across parallel lines yield a translation, and two reflections across non-parallel lines yield a rotation.

The three basic isometries can be represented by formulas in the Cartesian plane.

1. A translation T_P, where $P=(a,b)$ is given by
$$T_P(x,y)=(x+a,y+b)$$
2. A rotation R_θ with center the origin is given by
$$R_\theta(x,y)=(x\cos\theta-y\sin\theta, x\sin\theta+y\cos\theta)$$
3. A reflection across the x-axis is given by
$$J(x,y)=(x,-y)$$
A reflection across the y-axis is given by
$$J(x,y)=(-x,y)$$

Definition 12.3. A dilation is a type of transformation of the plane that does not preserve distances (and thus it is not a n isometry). Actually, a dilation scales up or down the figures on the plane and preserves the figure's angles.

Recall that to scale up/down (or zoom in/out) means that we will scale lengths, and that the effect on areas and volumes is a scaling that is different from the one used for lengths.

Remark 12.4. We can now re-define similarity of figures as $A\sim B$ if there is a dilation (maybe mixed with some isometries) that maps A onto B bijectively.

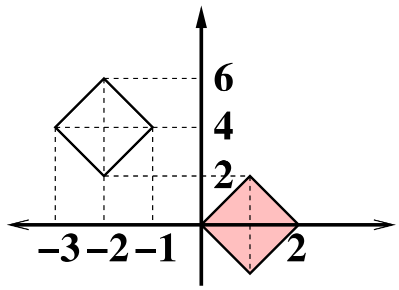
A dilation can be represented as a function. For example, if the dilation D reduces shapes to 50% of their size, then
$$D(x,y)=\frac{1}{2}(x,y)=\left(\frac{1}{2}x,\frac{1}{2}y\right)$$

The dilation D that triples the shapes is given by
$$D(x,y)=3(x,y)=(3x,3y)$$

Also, a transformation could dilate the plane under a different factor in the horizontal and vertical directions. For instance,
$$D(x,y)=\left(3x,\frac{1}{2}y\right)$$

Problems

12.1. In the following picture, the shaded rhombus on the right has been translated to the unshaded rhombus on the left. What is the vector used for this translation?



12.2. In the Cartesian plane there is a triangle ABC, where $A = (-3,2), B = (-1,4)$ and $C = (-2,6)$. By using a reflection across the x-axis we obtain a triangle $A'B'C'$. Give the coordinates of the vertices of this triangle.

12.3. (a) Find the 'formula' that describes a reflection across the line $y = a$, for some fixed real number a.
(b) Find the 'formula' that describes a reflection across the line $x = b$, for some fixed real number b.

12.4. (a) Compose a reflection across the x-axis with a reflection across the y-axis. What do you get?
(b) Compose the isometry found in (a) with a rotation in 45°. What do you get?
(c) Compose a reflection across the x-axis with a reflection across the line $y = 1$. What do you get?

Chapter 13
Probability

In order to find the probability of an event to occur we need to be aware of all the possibilities of the activity we are looking at. For example, if we want to know the probability of getting tails when flipping a coin, then all possible outcomes are "heads" and "tails". If we are throwing a dice then the possible end results are "1", "2", "3", "4", "5", and "6".

Of course, it is possible that if I throw a dice a gust of wind could blow it away and keep it floating forever in front of my eyes... but that is not a case that we will consider as a 'possible outcome'.

13.1 Simple probability

Whenever we want to compute the probability of *something* we will restrict our attention to the activity we have to observe to determine whether *something* occurs. This activity is called an experiment, and all the possible combinations of outcomes of the experiment are called events. Thus, we will always compute the probability of an event.

Definition 13.1. The probability of the event X to occur when we perform the experiment E is given by

$$P(X) = \frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}}$$

where an outcome is favorable if its occurrence means that X occurs.

Example 13.1. The probability of throwing a dice and getting a 3 is $1/6$ because there are 6 outcomes and only one of them (getting a 3) is favorable.

Similarly, the probability of throwing a dice and getting a prime number is $3/6$ because out of the 6 possible outcomes, exactly three of them are favorable; they are 'getting a 2', 'getting a 3' and 'getting a 5'.

Example 13.2. Assume you have a 52-card deck. The probability of choosing one card at random from the deck and get an ace is $4/52$, as getting any of the cards is an outcome (52 in total), but only four of those cards are aces. Hence, there are exactly 4 favorable outcomes.

Note that, since the number of favorable outcomes, for any experiment and event, will always be at most the total number of outcomes, then the probability of an event is always less or equal to 1. Moreover, if the event is impossible then its probability is 0 and if it is certain then its probability is 1.

Sometimes, counting the outcomes of an experiment is not possible. For example, if the experiment is to throw a javelin at the Olympics and the outcomes are all the possible distances the javelin travels before landing, then the outcomes are ALL real numbers between 0 and 100 meters (the WR is almost 100 meters)... this is infinitely many outcomes!

Example 13.3. Assuming all distances are equally probable to be reached. What is the probability of throwing a javelin and land it beyond 75 meters?

The total ‘number’ of outcomes in this case is represented by the 100 meters (from 0 to 100) where the javelin could land. The favorable outcomes are the last 25 meters (beyond the 75-meter mark) where we want to land the javelin. It follows that the probability is

$$P(X) = \frac{25}{100} = \frac{1}{4}$$

Example 13.4. My bag of markers has markers of many different colors: 12 of them are red, 7 are blue, 3 are purple, 20 black, and 1 is orange. What is the probability of, without looking, choosing a red marker?

13.2 Probability with multiple events

Let X and Y be two events, we know how to find the probability of them (previous section). Now we want to compute the probability of the events X-OR-Y, and X-AND-Y.

Note that if X is to throw a fair dice and get a 3, and Y is to throw a fair dice and get a 6 then X-OR-Y, which is to throw a dice and get a 3 or a 6, could be re-phrased as to throw a dice and get a multiple of 3. We get,

$$P(X-OR-Y) = \frac{2}{6} = \frac{1}{6} + \frac{1}{6} = P(X) + P(Y)$$

Nice!

However, if X is to throw a fair dice and get an even number, and Y is to throw a fair dice and get a prime number then X-OR-Y, which is to throw a dice and get a 2, 3, 4, 5 or 6, does not work as well as before

$$P(X-OR-Y) = \frac{5}{6} \neq \frac{3}{6} + \frac{3}{6} = P(X) + P(Y)$$

this is because X and Y have favorable events that are common to both (to get a 2), so we we need to subtract this repeated favorable event. Thus,

$$P(X-OR-Y) = \frac{5}{6} = \frac{3}{6} + \frac{3}{6} - \frac{1}{6} = P(X) + P(Y) - P(\text{common event})$$

Now we realize that the common event can be phrased as $X-AND-Y$, as we want both X and Y to occur simultaneously. With this, we have the following theorem.

Theorem 13.1. $P(X-OR-Y) = P(X) + P(Y) - P(X-AND-Y)$

Most of the times, this theorem is phrased in terms of unions and intersections. It is not so hard to see that the favorable outcomes of $X-OR-Y$ are the union of the favorable outcomes of X union the favorable outcomes of Y. Similarly, the favorable outcomes of $X-AND-Y$ are the union of the favorable outcomes of X intersection the favorable outcomes of Y. Hence,

Theorem 13.2. $P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$

Example 13.5. What is the probability of randomly picking a card out of a 52-card deck and get an ace or a card that is diamonds.

Using the previous theorem,

$$P(ace\ or\ diamond) = P(ace) + P(diamonds) - P(ace\ and\ diamond)$$

Since

$$P(ace) = \frac{4}{52} \qquad P(diamonds) = \frac{13}{52} \qquad P(ace\ and\ diamond) = \frac{1}{52}$$

then

$$P(ace\ or\ diamond) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{4}{13} \sim 0.3$$

Sometimes computing $P(X \cap Y)$ is not easy, as the events X and Y could be interlocked. For example, if we want to compute the probability of getting in your socks drawer and randomly picking a black sock and then a white sock. In this situation, the second tim we pick from the drawer then we have already taken something out of it, and thus the total number of outcomes has changed. In order to straight this up we will need to introduce the concepts of dependence and independence of events.

Definition 13.2. Two events x and y are said to be dependent if the outcomes of one of them changes if we assume that the other has occurred. In an obvious way independent events are defined.

Since having two events to hold simultaneously could be thought of as a complex event that consists of two events, then it is not so surprising to get that the probability of the complex event to occur is the product of the probability of the two simpler events to occur. But, the possibility of having dependent events yields the following theorem.

Theorem 13.3. *Let $P(Y/X)$ be the probability of Y occurring assuming that X occurred, then*

$$P(X \cap Y) = P(X)P(Y/X)$$

Example 13.6. Assume a drawer contains 7 red socks, 8 black socks and 10 white socks. What is the probability of randomly picking two socks, one after the other, and get two black socks?

We want to use the formula above, so we need to find $P(black)$ and $P(black/black)$. Since there are 25 socks in the drawer, then

$$P(black) = \frac{8}{25}$$

If we now assume that a black sock has been picked then there are only 24 socks left in the drawer, and only 7 black socks left. So,

$$P(black/black) = \frac{7}{24}$$

It follows that

$$P(2\ black\ socks) = \frac{8}{25}\frac{7}{24} = \frac{7}{75}$$

Example 13.7. Assume a drawer contains 7 red socks, 8 black socks and 10 white socks. What is the probability of randomly picking two socks, one after the other, and get two black of the same color?

We first notice that the probability we want to compute can be written as

$$P(2\ red\ socks\ OR\ 2\ black\ socks\ OR\ 2\ white\ socks)$$

Since, there are no common cases in these three situations (can't pick 2 socks of one color and 2 socks of a different color), then

$$P(2\ red\ OR\ 2\ black\ OR\ 2\ white) = P(2\ red) + P(2\ black) + P(2\ white)$$

We have already computed $P(2\ black\ socks)$ in exercise 13.6. In a similar way we get

$$P(2\ red\ socks) = \frac{7}{25}\frac{6}{24} = \frac{7}{100} \qquad P(2\ white\ socks) = \frac{10}{25}\frac{9}{24} = \frac{3}{20}$$

It follows that

$$P(2\ red\ OR\ 2\ black\ OR\ 2\ white) = \frac{7}{100} + \frac{7}{75} + \frac{3}{20} = \frac{47}{150}$$

Remark 13.1. The complement of an event X is the event (denoted $\overline{X}$) determined by having favorable outcomes the set of all possible outcomes that are not favorable outcomes of X.

Since $X \cup \overline{X}$ will cover the set of all possible outcomes, then

$$1 = P(X \cup \overline{X})$$

and thus $P(\overline{X}) = 1 - P(X)$.

13.3 Counting

We close this chapter with a set of important tools we will need to find complex probabilities. In the examples we have seen before, both the number of favorable outcomes and the total number of outcomes have been fairly easy to find. Now we will learn how to count sets that will not be that easy to count. We start with the fundamental principle of counting.

Theorem 13.4 (Fundamental principle of counting). *If a process P consists of k processes $P_1, P_2, \cdot, P_k$ performed back-to-back, and each process P_i can be done in n_i distinct ways, then the number of distinct ways the process P can be done is*

$$n = n_1 n_2 \cdots n_k$$

Example 13.8. We want to drive from Fresno to Denver. In our road trip we must go through Las Vegas. If there are three different roads connecting Fresno and Las Vegas, and there are five distinct roads from Las Vegas to Denver, then how many distinct roads are there from Fresno to Denver?

We consider the trip from Fresno to Denver as a process that consists of two other processes: Fresno-Las Vegas and Las Vegas-Denver. Since we know in how many ways these processes can be performed then we know that there are $3 \cdot 5 = 15$ roads joining Fresno and Denver.

Example 13.9. I own three pairs of shoes, two pairs of pants and ten t-shirts. In how many ways can I get dressed? Meaning how many different outfits shoes-pants-shirts can I assemble with what I have in my closet?

The process of assembling an outfit has three parts, choosing shoes, choosing pants and choosing a t-shirt. It follows that the number of possible distinct outfits is $3 \cdot 2 \cdot 10 = 60$.

Definition 13.3. Let n be a natural number, then $n!$, read n factorial is the product

$$n! = 1 \cdot 2 \cdots n$$

and, by convention, $0! = 1$.

Now we want to see in how many ways we can choose k things out of a set of n things. This process depends on whether or not it matters the order we choose things. Order is relevant as, for example, it is not the same to choose three representatives from a class of 30 students, than to choose a president, vice-president, and a treasurer from a class of 30 students. We are, in both cases, choosing three people out of 30 but one student being president or treasurer does make a difference, and thus the second case is essentially different from the first.

If order matters, then choosing k things out of n can be thought of as placing k objects in k (ordered) boxes. As the figure below shows, in the first box, we can place any of the n things, in the second box, we can place any of the $n-1$ things that are left, in the third box we can place any of the $n-2$ things left, etc. it will all stop at the k^{th} box, where we can place any of the $n-k+1$ remaining things.

n	$n-1$	$n-2$	$\cdots\cdots$	$n-k+1$

By using theorem 13.4 we now that the number of ways to choose k things out of n, when order matters is

$$n \cdot n-1 \cdot \cdots \cdot (n-k+1) = \frac{(n \cdot n-1 \cdot \cdots \cdot (n-k+1))((n-k) \cdots 2 \cdot 1)}{(n-k) \cdots 2 \cdot 1}$$
$$= \frac{n!}{(n-k)!}$$

Now, since choosing k things, order mattering, out of k things can be done in $k!$ ways (using formula above), then each bunch of k, things with no order mattering, yields $k!$ possible sets of with order mattering. It follows that the number of ways to choose k things out of n, when order does not matter is

$$\frac{n!}{k!(n-k)!}$$

which is also denoted $\binom{n}{k}$, read 'n choose k'.

Definition 13.4. A way to choose k distinct things out of n, when order matters, is called a permutation, the total number of these permutations is denoted $P(n,k)$. A way to choose k things out of n, when order does not matter, is called a combination, the total number of these permutations is denoted $C(n,k)$.

Summarizing.

Theorem 13.5. *Let n,k be natural numbers and $k \leq n$. Then*

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

and

$$P(n,k) = \frac{n!}{(n-k)!}$$

Example 13.10. What is the probability of choosing four cards off a 52-card deck so that the first is $A\diamondsuit$, the second is $2\clubsuit$, the third is $3\heartsuit$, and the fourth is $4\spadesuit$?

In this experiment, the possible outcomes are sets of four cards, chosen in order. It follows that the total number of outcomes is given by all possible ways to choose 4 cards out of the 52 in the deck, IN ORDER. So, the total number of outcomes is

$$P(52,4) = \frac{52!}{(52-4)!} = \frac{52!}{48!} = 52 \cdot 51 \cdot 50 \cdot 49 = 6,497,400$$

The favorable outcomes have to be found in the set of all favorable outcomes. Since we want exactly one of those cases to occur, the there is exactly one favorable outcome. Hence, the probability asked is

$$P = \frac{1}{6,497,400} = 0.00000015$$

which is a 0.000015% chance. Very low.

Example 13.11. What is the probability of choosing four cards off a 52-card deck so that the first is an A, the second is a 2, the third is a 3, and the fourth is a 4?

Note that in this problem the total number of outcomes is equal to the previous example. The favorable outcomes are different, though. In this case, the first card must be an ace, thus there are four cards that work. For the second, we have four more, either of the 2's, etc. It follows that the number of favorable outcomes is given by

4	4	4	4

which is 4^4. Hence, the probability asked is

$$P = \frac{4^4}{6,497,400} = 0.0000394$$

which is a 0.00394% chance. Very low, but not as low as in the previous example.

Example 13.12. What is the probability of choosing four cards off a 52-card deck simultaneously so that they are $A, 2, 3$ and 4?

Since in this case we are choosing four cards simultaneously, then order is not relevant. It follows that all the outcomes are sets of four cards out of the 52 in the deck, and that order does NOT matter. Hence, the total number of outcomes is (using computations done in the previous examples)

$$C(52,4) = \frac{1}{4!}\frac{52!}{(52-4)!} = \frac{6,497,400}{24} = 270,725$$

The number of favorable outcomes is exactly the same as in the previous exercise. Hence, the probability asked is

$$P = \frac{4^4}{270,725} = 0.0002364$$

which is a 0.02364% chance. Very low again.

Problems

13.1. I throw a fair dice with 20 sides (a $d20$). What is the probability of getting a prime number?

13.2. My bag of markers has markers of many different colors: 12 of them are red, 7 are blue, 3 are purple, 20 black, and 1 is orange. What is the probability of, without looking, choosing a red marker?

13.3. Consider a square with side 10 and its inscribed circle. What is the probability that if I randomly choose a point 'inside' the square, then the point is inside the circle?

13.4. Assume that the same number of people are born every day of the year (365 days). What is the probability of choosing somebody on the street at random and have that person's birthday to be in a month ending in "ary".

13.5. This is a classical problem, excellent to try in class.

What is the probability of two people in a class of 23 students to have the same birthday?

Chapter 14
Statistics

Statistics can be defined as the science that is used to analyze data. For many this is a subarea of mathematics and for many others it is a completely different science. We will start with simple data analysis and later we will see how to use data to test assumptions.

14.1 Analysis of data

In order to study data (in a statistical way) we need them to be expressed in terms of numbers. We cannot find, for example, the mean of the colors of the rainbow, as colors cannot be added, multiplied, etc. Hence, whenever data is given to us we will put them in a list $x_1, x_2, \cdots, x_n$, where all the x_i's are real numbers. Note that some of these x_i's may appear more than once in the list, and that since we are dealing with numbers here then we can always re-arrange the data in an increasing (or decreasing) order.

Definition 14.1. The number of times a value x_i appears in the list of data is called the frequency of x_i. Many times, the frequency of x_i is denoted f_i.

We want to learn about how the data obtained is distributed. Most of the times we want this to be able to make decisions. For instance, consider the following data

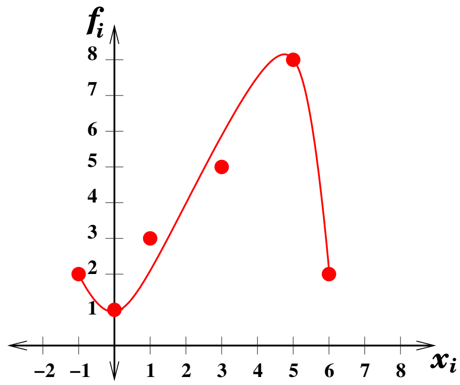
$$-1,-1,0,1,1,1,3,3,3,3,3,5,5,5,5,5,5,5,5,6,6$$

and assume it gives how early/late you arrive to the classroom (negative being early). Then by looking at the distribution of the data we know that you most of the times are getting at least three minutes late to class.

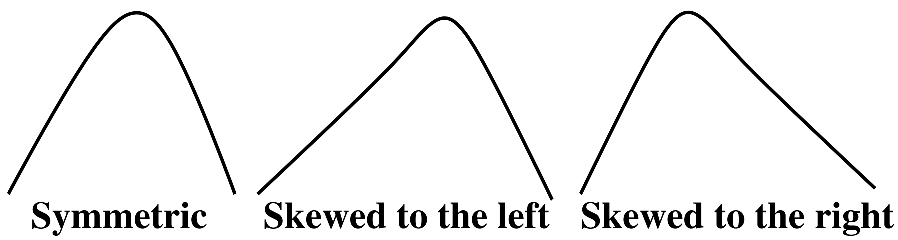
This way to present data is fine and good but it is not very easy to use when there are many values involved. In order to simplify this we will use ordered pairs (x_i, f_i), where x_i is a datum and f_i its frequency. In this case we consider the data given with no repetitions because each datum will be attached to its frequency. For instance, the following data given above can also be written as

$$(-1,2),(0,1),(1,3),(3,5),(5,8),(6,2)$$

Now it is obvious that these points can be put in the Cartesian plane (see chapter 10), where the x-axis is for the data values and the y-axis is for the frequencies. Note that these points could even be considered as points on a curve, as in the following picture, which is a graphic presentation of how the data was distributed.



Most of the times these graphs will have a ‘bell-shape’, that could be skew to one of the sides as the following pictures show.



The distribution of data given by a graph that looks like the symmetric graph above is called a *normal distribution*. We will later discuss these graphs again.

Let us start looking at ways to decide how the data is distributed. The main tools will be to determine what value ‘represents’ better the data given. We will define the three most used measures of central tendency next.

1. The *arithmetic mean* of $x_1, x_2, \cdots, x_n$ is
$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}$$
Some times the mean is denoted μ, this is mostly when the data is about population. Also, note that GPA’s is usually the arithmetic mean of the student’s grades.
2. The *mode* of $x_1, x_2, \cdots, x_n$ is the value that has the highest frequency. In case there are many values with highest frequency then all of them (or each one of them, if you prefer) are the mode of that sample of data.
3. The *median* of $x_1, x_2, \cdots, x_n$ is the value that is right in the middle of our increasing (or decreasing) list of values. In case there is no middle value (there are an even number of values) then the arithmetic mean of the two middle values is considered to be the median of the sample.

Example 14.1. Consider the data used in a previous example

$$-1, -1, 0, 1, 1, 1, 3, 3, 3, 3, 3, 5, 5, 5, 5, 5, 5, 5, 5, 6, 6$$

Then,

$$\begin{aligned} \bar{x} &= \frac{-1 + -1 + 0 + 1 + 1 + 1 + 3 + 3 + 3 + 3 + 3 + 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 + 6 + 6}{21} \\ &= \frac{68}{21} \\ &\sim 3.24 \end{aligned}$$

The mode is clearly 5 (it has frequency 8), and the median is 3.

I don't know what you think about computing arithmetic means but it seems to be too long, and it could get really long when the number of values given increases. So, note that

$$\begin{aligned}\bar{x} &= \frac{-1+-1+0+1+1+1+3+3+3+3+3+5+5+5+5+5+5+5+5+6+6}{21}\\ &= \frac{-1(2)+0(1)+1(3)+3(5)+5(8)+6(2)}{21}\end{aligned}$$

So, the arithmetic mean of $x_1, x_2, \cdots, x_n$ can also be computed by re-writing the data using ordered pairs (x_i, f_i) and then the numerator of $\bar{x}$ is obtained by adding all the products $x_i f_i$ for **distinct** values of x_i. The denominator or $\bar{x}$ stays the total number of values in the sample.

Example 14.2. Consider two companies, A and B, each with 21 people working in them. In company A everybody makes \$ $100,000$ a year, and in company B the owner makes \$ $1,000,000$ a year, ten employees make \$$100,000$ a year and other ten make \$ $10,000$ a year.

The arithmetic mean of the salaries in company A is

$$\frac{100,000(21)}{21} = 100,000$$

per year.

The arithmetic mean of the salaries in company B is

$$\frac{1,000,000(1)+100,000(10)+10,000(10)}{21} = 100,000$$

per year.

So, the arithmetic mean does not catch the difference of salaries between the two companies.

The median for company A and B is the same (\$ $100,000$ a year). The mode in company A is \$ $100,000$ a year, and the mode in company B is both \$ $100,000$ a year and \$ $10,000$ a year. Here some difference was caught but it would be very easy to manipulate the info to say that the mode of the salaries in company B is just \$ $100,000$ a year.... in which case these two companies would have no noticeable difference, but there is a big difference!!!

It is pretty clear now that we need to add ways to discriminate between data given to us, beyond looking at the mean, mode and median. We need to know how spread the data is. Note that in the previous example, one of the companies had all the data at one value (same salary), the other one had the data more spread, with one salary being very far from all the others. We will use the standard deviation and the variance as measures of dispersion.

Definition 14.2. The standard deviation of $x_1, x_2, \cdots, x_n$ is

$$S = \sqrt{\frac{\sum (x_i - \bar{x})^2}{n}}$$

this can also be written as

$$S = \sqrt{\frac{(x_1-\bar{x})^2+(x_2-\bar{x})^2+\cdots+(x_n-\bar{x})^2}{n}}$$

Sometimes, the letter σ is used to denote the standard deviation, but this is mostly used when the data is related to population.

Since the square root in the previous definition is a little too ugly, then we consider the following concept

Definition 14.3. The variance of $x_1, x_2, \cdots, x_n$ is the square of the standard deviation of the same data.

Example 14.3. Consider the data given in example 14.2. Let us find the standard deviation of the salaries of both companies.

For company A, the mean is $\bar{x} = 100,000$, and every value considered is also $100,000$, it follows that $x_i - \bar{x} = 0$ for all i, and thus $S_A = 0$.

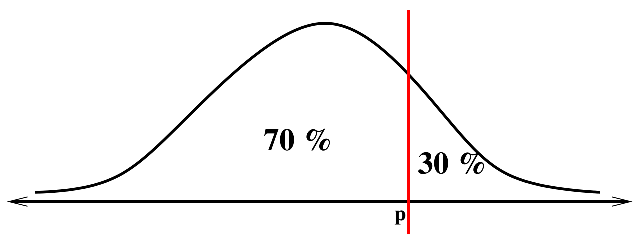
For company B, the mean is $\bar{x} = 100,000$, and many values are equal to $100,000$, but not all of them. So, we need to actually compute the variance

$$\begin{aligned} S_B^2 &= \frac{(1,000,000-100,000)^2+(100,000-100,000)^2(10)+(10,000-100,000)^2(10)}{21} \\ &= \frac{(900,000)^2+0+(-90,000)^2(10)}{21} \\ &= \frac{(900,000)^2+(90,000)^2(10)}{21} \\ &\sim 42428571430 \end{aligned}$$

and thus $S_B \sim 205,981.9687$.

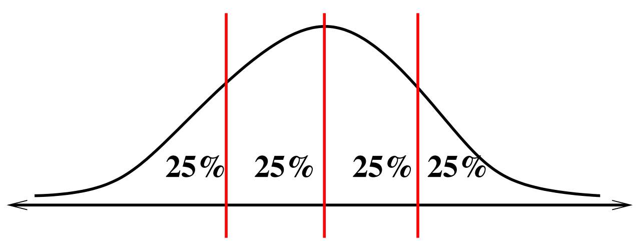
So, the standard deviation does capture that the salaries are not distributed in the same way for the two companies. In fact, most of the times the three measures of central tendency and two measures of dispersion we have learned about in this chapter give a pretty good idea on how the data is distributed. Trust them, these are the main tools you will need to understand and know how to use for your test!

Let us recall that the data given, when written as a set of ordered pairs (datum, frequency), could form a graph that most of the times looks like a bell, or a skewed one (do not assume these graphs always look like this). Now think of a vertical line that cuts the graph (not necessarily a bell) into two regions (red line in the picture below)



so that 70% of the area under the graph is to the left of the line, and of course 30% is to its right. This line is associated to a datum (labeled p in the picture). This value is called the 70^{th} percentile of the data represented by the graph.

More precisely, the k^{th} percentile of a set of numbers arranged in an increasing order is the value that has $k\%$ of the numbers below it... and $(100-k)\%$ above it. When the area below the graph (not necessarily a bell) is partitioned into four equally big parts by three values (associated to the three red lines)



then we have found the three quartiles of the data represented by the graph. These values are denoted lower quartile, median quartile and upper quartile (from left to right). In other words, the lower quartile is the 25^{th} percentile, the median quartile is the 50^{th} percentile, and the upper quartile is the 75^{th} percentile.

One of the reasons that all the pictures above have been bells, is because if the data given to us is normally distributed, then we can learn quite a few things about them For instance, if data is in a normal distribution, then the graph that represents the data is symmetric and this forces the mean, mode and median to be the same value, which is the 50^{th} percentile. Moreover, in this case the standard deviation S gives very important (although approximate) information about how the data is distributed. Specifically, about 68% of the data will be at distance at most S from the mean $\bar{x}$, about 95% of the data will be at distance at most $2S$ from $\bar{x}$, and about 99.7% of the data will be at distance at most $3S$ from $\bar{x}$. We can see this in the following picture

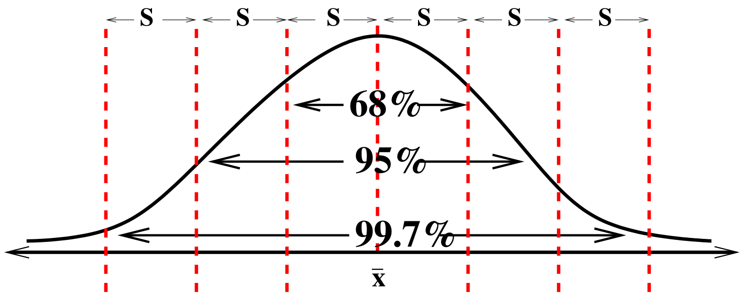


Fig. 14.1 Data distribution and distance to the mean.

14.2 Curve fitting

In order to understand how data behaves, we might want to see how they relate to another set of data, or with theoretical results. Also, sometimes one wants to see if there is any hint of two results being connected. For instance, we would like to know whether the percentage of population that can swim depends on the distance from people to the closest beach.

Sometimes we want to see if there is a *correlation* between two sets of data. In statistical terms a correlation coefficient is a number that measures how directly close (if at all) two sets of data relate to each other. In this case we are using the word 'directly' to mean a proportional relation between the two sets of data, this can be phrased in other words by saying that if we plotted points (x_i, y_i), where the x_i's are from one set of data and the y_i's are from the other and then the correlation coefficient would determine how close are these points from being on a straight line.

The correlation coefficient will always be a number between -1 and 1. Whenever it is zero will mean that the two sets of data are probably not on a linear relation. On the other hand, the closer the correlation coefficient is to ± 1 then the better the relation between the two sets of data is approximated by a line. Positive or negative coefficients would mean that the line that approximates the plotted points will have positive or negative slope. This number is found by computing

$$r = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sqrt{\sum(x-\bar{x})^2\sum(y-\bar{y})^2}}$$

Moreover, the line that is the 'closest' to the data is called the regression line, and the process to find it is called linear regression. The equation of this line is given by $y = a + bx$, where y is considered to be depending on x, and the numbers a and b are given by

$$b = \frac{\sum xy - n\bar{x}\,\bar{y}}{\sum x^2 - n\bar{x}^2} \qquad\qquad a = \frac{\sum y - b\sum x}{n}$$

As usual, n denotes the number of data.

Example 14.4. Assume that the following table shows the percentages of people who can swim living in a certain town, and the distance from the town to the nearest beach. We will consider the number of swimmers to depend on the distance to a beach.

	distance (miles)	% swimmers
Town 1	100	56
Town 2	0	82
Town 3	600	50
Town 4	500	70
Town 5	60	76
Town 6	300	62

Then we have the point (x_1, y_1) attached to Town i. In order to find the correlation coefficient we need to find $\bar{x}$ and $\bar{y}$. We get,

$$\bar{x} = \frac{100+0+600+500+60+300}{6} = 260$$
$$\bar{y} = \frac{56+82+50+70+76+62}{6} = 66$$

It follows that

$$r = -0.607751 \qquad a = -0.0298 \qquad b = 73.7662$$

Computations are long and it is very easy to make mistakes doing them. So, we will do them using a calculator (see next section).

Now, what does the regression line $y = ax + b$ we have just found mean to us? Most of the times it is used to *approximate* some info about the way the data increases/decreases. For instance, in example 14.4 the value of b is the y-intercept of the line, and thus it is the value that should correspond to $x = 0$, which could be interpreted as the percentage of people that should be expected to swim for a town by a beach (distance 0 miles to the closest beach). Note that $b \sim 74$ is consistent with the data we have for Town 2. Also, looking at a (the slope of the line) we could say that every mile getting away from a beach should decrease the percentage of swimmers by approximately 0.3%. However, it is important to note that the correlation coefficient is $r \sim -0.6$, which is not that small, and thus the approximation could be not that good for certain values.

Example 14.5. With the data given in example 14.4. What should be the percentage of swimmers living in a town that is 200 miles from the closest beach?

Since the data is approximated by $y = -0.0298x + 73.766$, then the number of swimmers, given by y, that corresponds to $x = 200$ is $y = -0.0298 \cdot 200 + 73.766 = 67.806$, which is fairly consistent with the info we have from Town 1 and Town 6.

Note that in the previous example we imply that we can guess what the values of one set of data could be obtained using the line that approximates the raw data. The question at this point is obvious: How good is this approximation? can we use it to generate new data? These questions are answered in the following section.

14.3 Hypothesis testing

In this section we will learn how to use statistics to investigate the validity of an assumption by studying data that is related to such assumption. Most of the times we will have a fresh set of data that needs to be compared with old data, or we will have data that we would like to compare with theoretical values.

We will call *null hypothesis* to the hypothesis/assumption/statement that implies no change. So, the null hypothesis should always include the word ‘equal’ (or the equal sign). The final decision we will make will be either that the null hypothesis is rejected or that there is no sufficient evidence to reject it.

The negation of the null hypothesis is called the *alternative hypothesis*. We will denote the null hypothesis by H_0 and the alternative hypothesis by H_1.

Example 14.6. Assume that the scores in a math class (CSET II, for instance) ten years ago were 43% got A’s, 29% were B’s, 14% were C’s and there was also a 14% of D’s.

Last year out of 500 people taking the test, there were 199 A’s, 153 B’s, 70 C’s and 78 D’s.

We want to show that the current distribution of the data is different from that one ten years ago. Hence,

H_0 = the current data distribution is the same as the distribution ten years ago.

H_1 = the current data distribution differs from the distribution ten years ago.

Now note that the mean for the set of old data cannot be computed as we have done before because we do not know how many values were considered to find those percentages. So, what we do is that we let 100 to be the total number of data, and the percentages will dictate the frequencies of the values. Also, since we cannot compute the mean of letters then we need to transform the grades into numbers (we use the standard scale from 1 to 4 for this). Hence,

$$\begin{aligned}
\bar{x}_{old} &= \frac{43 \cdot 4 + 29 \cdot 3 + 14 \cdot 2 + 14 \cdot 1}{100} \\
&= \frac{172 + 87 + 28 + 14}{100} \\
&= \frac{301}{100} = 3.01
\end{aligned}$$

$$\begin{aligned}
\bar{x}_{new} &= \frac{199 \cdot 4 + 153 \cdot 3 + 70 \cdot 2 + 78 \cdot 1}{500} \\
&= \frac{796 + 459 + 140 + 78}{500} \\
&= \frac{1473}{500} = 2.946
\end{aligned}$$

We can see that the means are fairly close, but recall that sometimes the mean is not enough to determine the distribution of data. So, let is compute the standard deviations.

$$\begin{aligned}
S_{old}^2 &= \frac{43 \cdot (4 - 3.01)^2 + 29 \cdot (3 - 3.01)^2 + 14 \cdot (2 - 3.01)^2 + 14 \cdot (1 - 3.01)^2}{100} \\
&= \frac{43 \cdot (0.99)^2 + 29 \cdot (-0.01)^2 + 14 \cdot (-1.01)^2 + 14 \cdot (1 - 2.01)^2}{100} \\
&= \frac{42.1443 + 0.0029 + 14.2814 + 56.5614}{100} = 1.1299
\end{aligned}$$

So, $S_{old} = \sqrt{1.1299} \sim 1.063$.

$$\begin{aligned}
S_{new}^2 &= \frac{199 \cdot (4 - 2.946)^2 + 153 \cdot (3 - 2.946)^2 + 70 \cdot (2 - 2.946)^2 + 78 \cdot (1 - 2.946)^2}{500} \\
&= \frac{199 \cdot (1.054)^2 + 153 \cdot (0.054)^2 + 70 \cdot (-0.946)^2 + 78 \cdot (-1.946)^2}{500} \\
&= \frac{221.07228 + 0.446148 + 62.64412 + 295.37945}{500} = 1.159084
\end{aligned}$$

So, $S_{new} = \sqrt{1.159084} \sim 1.076$.
Just like with the means, the standard deviations are pretty close. Should we conclude that these two sets of data are consistent with each other?

What we did above could be used to compare the two sets of data. However, it seems long and cumbersome. Next we will learn about tests that will be used to determine how consistent two sets of data are. In order to do this we need to be aware of a couple of details:
(a) Since data can come in different amounts and orders of magnitude, we will need to standardize the values given to us. By doing this we will be able to display the data in a generic way, and thus we will be able to use pre-constructed tables (see appendix) that will do most of the computational work for us!!
(b) For most purposes, old data and theoretical data would be considered as the known and accepted information against which we want to compare our new data.
(c) The percentage of significance is a number that we will use to determine how much is the most we will accept of difference between the two sets of data. Most of the times this percentage is 5%. In this case we will deny the

null hypothesis only if the difference between the two sets of data is more than 90% (or 95%, depending on the test).

We will test sets of data for consistency using two types of tests: the z-score and the χ^2-test.
In both tests, we will reject the null hypothesis if the test score that represents our data falls farther than a certain critical value. This critical value depends on the test used and on the information given in the problem.

z-score

For the one-sample z-score we will assume our data (new and old) to be distributed normally (in a bell shape). In this case we will use a formula that will standardize the data and give us a number (called the z-score). In simple terms, this score will tell us whether our data is 'too far away' from the (standardize) old mean. All this must be done by looking at table 1 in the appendix, and taking into consideration the percentage of significance we want to give to the test.

Example 14.7. Assume the z-score of the sample is $z = 2.34$. This would mean that the new data is 2.34 (standardize) standard deviations from the (standardize) old mean.
Now we look at table 1 in the appendix and see that for $z = 2.34$ the area (on the right half of the normal bell curve) is given by the table as $A = .49$. This means that the new data is in the farthest $100 - 2 \cdot 49 = 2\%$ of the old data, that is pretty far away! (most of the times the acceptable percentage will be 5%). Hence, in such case the null hypothesis would be rejected.

The z-score will allow us to compare two sets of data. As mentioned before, in most cases, one set is our 'control' set; either known data from the past or data that is generated out of theoretical work (linear regression, for instance). The second set is, most of the times, recent data obtained from some type of poll, study, etc.

Assume you know the following:
(a) $\bar{x} =$ new data mean,
(b) $\mu =$ theoretical mean or old mean,
(c) $s =$ new data standard deviation,
(d) $n =$ number of new data.
then the z-score is

$$z = \frac{\bar{x} - \mu}{s / \sqrt{n}}$$

where the value

$$\frac{s}{\sqrt{n}}$$

is called the standard error of the (new data) mean.

The z-score obtained can then be looked in table 1 in the appendix to see what we can say about the null hypothesis. Note that a z-score of 1.96 yields an area under one side of the bell of .475, which means an area of $2 \cdot 0.475 = .95$ overall (a 95% of it all), that means that any z-score larger than 1.96 (or less that -1.96) implies that our data falls below the 5% of significance, and thus the null hypothesis would be rejected.

Example 14.8. With the same data of example 14.6. We want to show that the new scores are statistically different from the old ones with a 5% of significance. So, we need to take H_0 to be 'the current data distribution is the same as the distribution ten years ago' (note that word equal, although not used, is implied).

Note that we are considering two situations at the same time here: the new scores could be higher or lower than the old ones.

From example 14.6 we know
(i) $\bar{x} = 2.946$ (new data mean)
(ii) $\mu = 3.01$ (old data mean)
(iii) $s = 1.076$ (new standard deviation)

(iv) $n = 500$ (number of new data)
then the standard error is

$$\frac{1.076}{\sqrt{500}} \sim 0.048$$

and thus the z-score is

$$z = \frac{2.946 - 3.01}{0.048} \sim -1.33$$

We look at table 1 in the appendix and we see that $z = 1.33$ corresponds to (one-sided) area $A = .408$, which means a double-sided area of .816, which is much closer than the .95 that the 5% of significance asks for. It follows that the null hypothesis cannot be rejected, and thus we can say that the new data is consistent with the old one.

Remark 14.1. If the hypothesis we wanted to show true would have been that the new data was 'larger' than the old one, then, since the z-score has $(\bar{x} - \mu)$ in its numerator then we would have been interested only in positive values for the z-score, and thus the area to consider should be only on the right hand side of the bell.

χ^2-test

There are two χ^2-test to consider. One of them is similar to the z-score test, but in this one we use a different table (table 2 in the appendix). As before we want to test new data by comparing it with old data (or theoretical data). We first compute the 'χ^2-score',

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2}$$

where s^2 = new data variance. σ^2 = theoretical, or old, variance, and n = number of items in the new data set.

As usual, the data must be analyzed with certain level of significance, this number must be found in the first row of table 2. For instance a 5% of significance will place us at the 0.05 entry (third row). Next we must find the number of degrees of freedom of our problem. This value is given by $n-1$, where n is the number of values in the new data set. Once this is known, this number must be located in the extreme left column of table 2. The intersection of the appropriate row and column will give us the critical value, which we will compare to the χ^2-score to see if we are able to reject the null hypothesis. Specifically, as we did with the z-score, if we get a χ^2-score that is beyond the critical value, then we will reject the null-hypothesis.

Example 14.9. Just as we did in example 14.8. With the same data of example 14.6. We want to show that the new scores are statistically different from the old ones with a 5% of significance. So, we need to take H_0 to be 'the current data distribution is the same as the distribution ten years ago'.

From example 14.6 we know
(i) $\sigma = 1.063$ (old standard deviation)
(ii) $s = 1.076$ (new standard deviation)
(iii) $n = 500$ (number of new data)

But this is too many data for our table! So, we will 'standardize' them a little. Since there are 500 data and our table gets to 100 degrees of freedom, then we will divide everything by 5, without really changing the variances.

Then the χ^2-score is

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2} = \frac{(100-1)(1.076)^2}{(1.063)^2} = 101.4362553309$$

Since the critical value corresponding to 5% significance and 99 degrees of freedom is 113.145 (found in table 2) then our χ^2-score is lower than the critical value, and thus we cannot reject the null hypothesis.

The second χ^2-test is used when more details of the data are known. Above we only knew information about the data but nor the data itself. In case that all the data is known, let us say $\{x_1, x_2, \cdots, x_n\}$ and $\{y_1, y_2, \cdots, y_n\}$, where x_i and y_i are the outputs of the same experiment but the x_i's are new data and the y_i's are old/theoretical data. In this setting:

$$\chi^2 = \sum \frac{(x_i - y_i)^2}{y_i}$$

After obtaining this number we can use table 2 as we did with the χ^2-score.

Remark 14.2. The second χ^2-test is more important than the first. You should use this one instead of the other unless you cannot perform it because of lack of data.

14.4 Calculator Use

Most calculator work in a similar way. Out of the ones that you are allowed to use in the CSET, we will go over how to use the TI-83 (or similar) and the HP 9g (or similar).

14.4.1 TI-83

In order to ask a calculator to do the long calculations needed in this test we first need to be able to create lists of data. I am using a TI-83 Plus, slight differences may exist with other calculators.

Press [STAT] to get to a window that says EDIT - CALC - TEST at the top. Move the cursor if necessary to get EDIT shaded, then press [ENTER]. Now you can see lists of data, if they are not empty you can delete their contents by getting the cursor on the header of the list and press [CLEAR].

Once the lists are emptied you can input your data. Just move the cursor over L1, for instance, and drop the cursor to the next cell, add the datum and press [ENTER] to input the next datum. When you are done inputing data move your cursor to another list, if you want to create another list, quit by pressing [2^{nd}] [QUIT], or just plain start a new (statistics) process without quitting by pressing [STAT]. Note that most of the times when you create lists, you will create lists with the same number of data, also, please be sure that if you have data that are related, then they are in the same order when they are inputed. For instance, for the data in example 14.4, you want to see the data in your calculator to be just like it appears on the table given above.

When looking for correlation coefficients and/or linear regression, this is a 'trick' that makes things a little easier, as it will give you r, a and b in the same window.

Press [2^{nd}] [CATALOG] (above 0) and move the cursor down until you reach DiagnosticOn, press [ENTER] twice (you want to get 'DiagnosticOn Done').

To start finding r, a and b you need to press [STAT], move the cursor to CALC and press [4] (or just press [ENTER] after you moved the cursor to LinReg(ax+b)). Now you need to select the lists of data you want to compare. Press [2^{nd}] [LIST] and then select the list that contains the data for the independent data (the one that will be x in the line you are finding). Now you should be back to the LinReg window, press [,] to now add a second list just as we did before, this list is the variable that depends on the data in the previous list (this data will be represented by y in the line you are finding). Finally press [,] again and press [VARS], move the cursor to Y-VARS, press [ENTER] and select Y1. Now, you are back at the LinReg window for a last time. Press [ENTER] to get r, a and b.

If at this point you want to graph the data and the regression line you could press [Y=] and then [GRAPH]. In case the data seems not to appear, or it is too small, then you will need to press [WINDOW] and carefully input the ranges for the variables x and y so all the data given fits in these ranges. For instance, for the info given in example 14.4 we would like to set Xmin =-100, Xmax=700, the scale to Xscl=100 (how far apart the dashes in the x-axis would be), and for the y's we would want Ymin =0, Ymax=100 and Yscl=10.

In order to find the standard deviation, mean, etc of a list of data, you first have to create a list as explained above, then you have to press STAT, move the cursor to CALC and choose 1-Var Stats, you should now be looking at a window that says “1-Var Stats”. Now you press LIST to choose the list that contains your data, select it, and when you get to the next window press ENTER. The mean is $\bar{x}$, the standard deviation is σx, and the median is Med.

14.4.2 HP 9g

This one works slightly different, as you have to make the list of data when you have already chosen what computations you want to make.

In order to find the correlation coefficient and the regression line you need to start by pressing Mode, then choose 1Stat and press ENTER, at this point you are in statistics mode, and thus non-statistical calculations will not work as usual. After you have pressed ENTER you need to select Reg, press ENTER, select Lin and press ENTER again. Now you need to input the data. Press DATA and select Data Input. Now you will input data in pairs, the x and y that are linked are inputted right after the other with pressing ▼ in between values. For instance, the data in example 14.4 is inputted as

100 ▼ 56 ▼ 0 ▼ 82 ▼ 600 ▼ 50 ▼ 500 ▼ 70 ▼ 60 ▼ 76 ▼ 300 ▼ 62

Do not press ▼ after the last value.

Finally, you press 2^{nd} STATVAR and by placing the cursor on a, b or r you will see their value on the bottom right corner of the screen. The correlation coefficient is r and the regression line is $y = bx + a$.

Appendix: Tables.

In the following table, the columns under an A shows the area under the bell curve that is between $z = 0$ and the (positive) value of z that is by it. Some times this table is called a table of z-scores.

Note that since the bell is symmetric then we will not give the table for the negative values of z.

Table 1: z-scores table

z	A	z	A	z	A	z	A	z	A	z	A	z	A
.00	.000	.48	.184	.96	.331	1.44	.425	1.92	.473	2.40	.492	2.88	.498
.01	.004	.49	.188	.97	.334	1.45	.426	1.93	.473	2.41	.492	2.89	.498
.02	.008	.50	.191	.98	.336	1.46	.428	1.94	.474	2.42	.492	2.90	.498
.03	.012	.51	.195	0.99	.339	1.47	.429	1.95	.474	2.43	.492	2.91	.498
.04	.016	.52	.198	1.00	.341	1.48	.431	1.96	.475	2.44	.493	2.92	.498
.05	.020	.53	.202	1.01	.344	1.49	.432	1.97	.476	2.45	.493	2.93	.498
.06	.024	.54	.205	1.02	.346	1.50	.433	1.98	.476	2.46	.493	2.94	.498
.07	.028	.55	.209	1.03	.348	1.51	.434	1.99	.477	2.47	.493	2.95	.498
.08	.032	.56	.212	1.04	.351	1.52	.436	2.00	.477	2.48	.493	2.96	.498
.09	.036	.57	.216	1.05	.353	1.53	.437	2.01	.478	2.49	.494	2.97	.499
.10	.040	.58	.219	1.06	.355	1.54	.438	2.02	.478	2.50	.494	2.98	.499
.11	.044	.59	.222	1.07	.358	1.55	.439	2.03	.479	2.51	.494	2.99	.499
.12	.048	.60	.226	1.08	.360	1.56	.441	2.04	.479	2.52	.494	3.00	.499
.13	.052	.61	.229	1.09	.362	1.57	.442	2.05	.480	2.53	.494	3.01	.499
.14	.056	.62	.232	1.10	.364	1.58	.443	2.06	.480	2.54	.494	3.02	.499
.15	.060	.63	.236	1.11	.367	1.59	.444	2.07	.481	2.55	.495	3.03	.499
.16	.064	.64	.239	1.12	.369	1.60	.445	2.08	.481	2.56	.495	3.04	.499
.17	.067	.65	.242	1.13	.371	1.61	.446	2.09	.482	2.57	.495	3.05	.499
.18	.071	.66	.245	1.14	.373	1.62	.447	2.10	.482	2.58	.495	3.06	.499
.19	.075	.67	.249	1.15	.375	1.63	.448	2.11	.483	2.59	.495	3.07	.499
.20	.079	.68	.252	1.16	.377	1.64	.449	2.12	.483	2.60	.495	3.08	.499
.21	.083	.69	.255	1.17	.379	1.65	.451	2.13	.483	2.61	.495	3.09	.499
.22	.087	.70	.258	1.18	.381	1.66	.452	2.14	.484	2.62	.496	3.10	.499
.23	.091	.71	.261	1.19	.383	1.67	.453	2.15	.484	2.63	.496	3.11	.499
.24	.095	.72	.264	1.20	.385	1.68	.454	2.16	.485	2.64	.496	3.12	.499
.25	.099	.73	.267	1.21	.387	1.69	.454	2.17	.485	2.65	.496	3.13	.499
.26	.103	.74	.270	1.22	.389	1.70	.455	2.18	.485	2.66	.496	3.14	.499
.27	.106	.75	.273	1.23	.391	1.71	.456	2.19	.486	2.67	.496	3.15	.499
.28	.110	.76	.276	1.24	.393	1.72	.457	2.20	.486	2.68	.496	3.16	.499
.29	.114	.77	.279	1.25	.394	1.73	.458	2.21	.486	2.69	.496	3.17	.499
.30	.118	.78	.282	1.26	.396	1.74	.459	2.22	.487	2.70	.497	3.18	.499
.31	.122	.79	.285	1.27	.398	1.75	.460	2.23	.487	2.71	.497	3.19	.499
.32	.126	.80	.288	1.28	.400	1.76	.461	2.24	.487	2.72	.497	3.20	.499
.33	.129	.81	.291	1.29	.401	1.77	.462	2.25	.488	2.73	.497	3.21	.499
.34	.133	.82	.294	1.30	.403	1.78	.462	2.26	.488	2.74	.497	3.22	.499
.35	.137	.83	.297	1.31	.405	1.79	.463	2.27	.488	2.75	.497	3.23	.499
.36	.141	.84	.300	1.32	.407	1.80	.464	2.28	.489	2.76	.497	3.24	.499
.37	.144	.85	.302	1.33	.408	1.81	.465	2.29	.489	2.77	.497	3.25	.499
.38	.148	.86	.305	1.34	.410	1.82	.466	2.30	.489	2.78	.497	3.26	.499
.39	.152	.87	.308	1.35	.411	1.83	.466	2.31	.490	2.79	.497	3.27	.499
.40	.155	.88	.311	1.36	.413	1.84	.467	2.32	.490	2.80	.497	3.28	.499
.41	.159	.89	.313	1.37	.415	1.85	.468	2.33	.490	2.81	.498	3.29	.499
.42	.163	.90	.316	1.38	.416	1.86	.469	2.34	.490	2.82	.498	3.30	.500
.43	.166	.91	.319	1.39	.418	1.87	.469	2.35	.491	2.83	.498	3.31	.500
.44	.170	.92	.321	1.40	.419	1.88	.470	2.36	.491	2.84	.498	3.32	.500
.45	.174	.93	.324	1.41	.421	1.89	.471	2.37	.491	2.85	.498	3.33	.500
.46	.177	.94	.326	1.42	.422	1.90	.471	2.38	.491	2.86	.498	3.34	.500
.47	.181	.95	.329	1.43	.424	1.91	.472	2.39	.492	2.87	.498	3.35	.500

In the following table the χ^2 values corresponding to the probability of exceeding the critical value, or level of significance (top row) and the degrees of freedom (left column).

Table 2: χ^2-test table

df	0.995	0.99	0.975	0.95	0.90	0.10	0.05	0.025	0.01	0.005
1	---	---	0.001	0.004	0.016	2.706	3.841	5.024	6.635	7.879
2	0.010	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210	10.597
3	0.072	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345	12.838
4	0.207	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277	14.860
5	0.412	0.554	0.831	1.145	1.610	9.236	11.070	12.833	15.086	16.750
6	0.676	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812	18.548
7	0.989	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	3.490	13.362	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209	25.188
11	2.603	3.053	3.816	4.575	5.578	17.275	19.675	21.920	24.725	26.757
12	3.074	3.571	4.404	5.226	6.304	18.549	21.026	23.337	26.217	28.300
13	3.565	4.107	5.009	5.892	7.042	19.812	22.362	24.736	27.688	29.819
14	4.075	4.660	5.629	6.571	7.790	21.064	23.685	26.119	29.141	31.319
15	4.601	5.229	6.262	7.261	8.547	22.307	24.996	27.488	30.578	32.801
16	5.142	5.812	6.908	7.962	9.312	23.542	26.296	28.845	32.000	34.267
17	5.697	6.408	7.564	8.672	10.085	24.769	27.587	30.191	33.409	35.718
18	6.265	7.015	8.231	9.390	10.865	25.989	28.869	31.526	34.805	37.156
19	6.844	7.633	8.907	10.117	11.651	27.204	30.144	32.852	36.191	38.582
20	7.434	8.260	9.591	10.851	12.443	28.412	31.410	34.170	37.566	39.997
21	8.034	8.897	10.283	11.591	13.240	29.615	32.671	35.479	38.932	41.401
22	8.643	9.542	10.982	12.338	14.041	30.813	33.924	36.781	40.289	42.796
23	9.260	10.196	11.689	13.091	14.848	32.007	35.172	38.076	41.638	44.181
24	9.886	10.856	12.401	13.848	15.659	33.196	36.415	39.364	42.980	45.559
25	10.520	11.524	13.120	14.611	16.473	34.382	37.652	40.646	44.314	46.928
26	11.160	12.198	13.844	15.379	17.292	35.563	38.885	41.923	45.642	48.290
27	11.808	12.879	14.573	16.151	18.114	36.741	40.113	43.195	46.963	49.645
28	12.461	13.565	15.308	16.928	18.939	37.916	41.337	44.461	48.278	50.993
29	13.121	14.256	16.047	17.708	19.768	39.087	42.557	45.722	49.588	52.336
30	13.787	14.953	16.791	18.493	20.599	40.256	43.773	46.979	50.892	53.672
40	20.707	22.164	24.433	26.509	29.051	51.805	55.758	59.342	63.691	66.766
50	27.991	29.707	32.357	34.764	37.689	63.167	67.505	71.420	76.154	79.490
60	35.534	37.485	40.482	43.188	46.459	74.397	79.082	83.298	88.379	91.952
70	43.275	45.442	48.758	51.739	55.329	85.527	90.531	95.023	100.425	104.215
80	51.172	53.540	57.153	60.391	64.278	96.578	101.879	106.629	112.329	116.321
90	59.196	61.754	65.647	69.126	73.291	107.565	113.145	118.136	124.116	128.299
100	67.328	70.065	74.222	77.929	82.358	118.498	124.342	129.561	135.807	140.169

Problems

14.1. Find the standard deviation, variance, lower quartile, median quartile and upper quartile of the data given in example 14.1.

14.2. Explain/prove that if data is in a normal distribution then its mean, mode and median coincide.

14.3. Using the data given in example 14.1, and problem 14.1 find the 2.5^{th} percentile, the 16^{th} percentile and two values so that 81.5% of the data is between them.

Solutions

Chapter 1

1.1 By contradiction. Assume that a triangle has more than one right angle, then the sum of the angles of the triangle is going to be more than 180°, which is impossible! It follows that a triangle can have at most one right angle.

1.2 A circle centered at C.

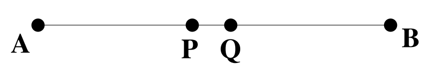
1.3 Let $\overline{AB}$ be a segment of length 5. Draw a circle with radius 2 centered at A and another with radius 4 centered at B. Since $2+4>5$ the these circles will intersect. Take one of the points of intersection and call it C. Join A with C and B with C, $\triangle ABC$ has sides as asked.

1.4 It is impossible because $2+4<7$ and thus, following the construction in the previous problem, the two circles drawn will not intersect.

Chapter 2

2.1 We first notice that both P and Q are collinear with A and B, since there is a unique line joining two points (A and B in this case) then P and Q must be on the line $\overleftrightarrow{AB}$.

Let $2d$ be the distance between A and B. In the picture below, we can see that d is the distance from P to A and also from Q to A, this forces the distance from P to Q to be zero. Contradiction.



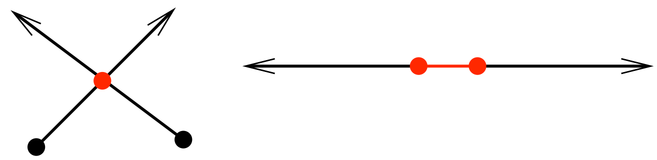
2.2 Given a segment $\overline{AB}$, let C be its midpoint, and let $\overleftrightarrow{CD}$ and $\overleftrightarrow{CE}$ be two distinct perpendicular bisectors of $\overline{AB}$, this contradicts the fact that these two lines will have a nonzero angle (remark 2.3) formed by them at C.

2.3 Fix an angle $\angle ABC$ and let $\overrightarrow{BD}$ and $\overrightarrow{BE}$ be two distinct angle bisectors of $\angle ABC$. Since these two rays share their initial point, then they form an angle, which by remark 2.3 it has positive measure. This is a contradiction, as assuming WLOG that $\overrightarrow{BD}$ is an angle bisector forces that $\overrightarrow{BE}$ is not a bisector of $\angle ABC$.

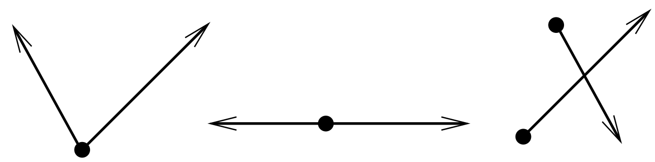
2.4 There are three different cases. If the four points are collinear then there is just one line, if three of them are collinear and the fourth is not, then there are 4 lines; the one containing three points and the ones from the non-collinear point to the three on a line. If at most 2 points are on a line (forming something like a square) then there are 6 lines; the four sides of the 'square' and the two diagonals.

There are no other cases, as any two points are always on a line.

2.5 The intersection of two rays might be: empty (parallel rays, for example), a ray (in case one ray is 'contained' in another), a point, or a segment, as in the following pictures.



Their union could be an angle, straight line, a cross-like shape, as in the figures below, but it could also be two disjoint rays, or just one ray (in case one of the rays is 'contained' in the other).



2.6 The ray $\overrightarrow{LK}$.

2.7 The following pictures show all the possible lines one can draw using those ten points. The first one shows 15 lines and the second the other 5.

2.8 Let us think that an angle is formed by sweeping, counterclockwise, a ray from its 'starting' ray to its 'final' ray. Hence, for each ray in the figure there are exactly three positive angles 'starting' at that ray (positive and less than 360°) and ending at one of the other three rays. Since there are four rays, then there are exactly $3 \cdot 4 = 12$ angles (positive and less than 360°) in the figure.

2.9 $0^\circ = 0$ radians, $30^\circ = \pi/6$ radians, $60^\circ = \pi/3$ radians, $90^\circ = \pi/2$ radians, $270^\circ = 3\pi/2$ radians, and $360^\circ = 2\pi$ radians.

2.10 $P = 2.5$ and $Q = 5$.

2.11 Let x be the measure of the angle we are looking for. We know that its supplement is $180^\circ - x$. The information given yields $x = 3(180^\circ - x)$, which implies $x = 135^\circ$. The angle is too large to have a complement.

2.12 Using the vertical angle theorem we get that the $\angle ABC = \alpha + \beta$. Since l is an angle bisector, then $\alpha = \beta$.

Chapter 3

3.1

3.2

3.3 (a) Since C is a point not on $\overleftrightarrow{AB}$ then Playfair's axiom assures the existence of a line through C that is parallel to $\overleftrightarrow{AB}$.
(b) Note that at C we have six angles, the three 'above' l_1 are equal to β, γ and α (respectively, from left to right) because of $l_1 || l_2$, proposition 29 (once with transversal $\overleftrightarrow{AC}$ and anther with transversal $\overleftrightarrow{BC}$), and the vertical angle theorem. It follows that $\beta + \gamma + \alpha = 180^\circ$, and thus the angle-sum of the triangle is also 180°.

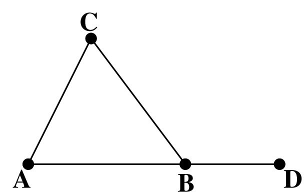
3.4 This is an activity you should do, so there is no solution.

3.5 This is an activity you should do, so there is no solution. However, after repeating the activity many times you probably will figure that the larger the triangle, the larger the sum of the angles of the triangle. In fact, in a sphere with radius r the following theorem holds
Theorem (Girard). Let $\triangle$ be a spherical triangle with interior angles α, β and γ, then its area is

$$A_{\triangle} = r^2 (\alpha + \beta + \gamma - \pi)$$

Note that this theorem confirms that the larger the angle-sum of the triangle, the larger the area of it.

3.6 Consider the figure



It is clear that $\angle ABC + \angle DBC = 180°$. Since we are in Euclidean geometry, then $\angle ABC + \angle CAB + \angle BCA = 180°$. It follows that $\angle CAB + \angle BCA = \angle DBC$.

On the other hand, if we were working in non-Euclidean geometry then $\angle ABC + \angle CAB + \angle BCA \neq 180°$, and thus $\angle CAB + \angle BCA \neq \angle DBC$.

3.7 The bear is white, as the man's house must be at the North pole. The North pole and the South pole are the only places on Earth (a sphere) where one can have an equilateral triangle as described in the problem.

3.8 If a triangle has angle sum equal to 200° then the triangle lives in an elliptic geometry, it follows that there are no parallel lines. So, the answer is zero.

If a triangle has angle sum equal to 100° then the triangle lives in a hyperbolic geometry, it follows that there are infinitely many parallel lines to ℓ through P.

Chapter 4

4.1

4.2

4.3 (a) True. A square with side a and a square with side b are similar with ratio $a \div b$.
(b) True, as the area of a square determines the length of a side, and thus we get congruency.
(c) False. Consider the following two rectangles: the first is a square with side 4, the second is a rectangle with sides measuring 1 or 4. Same area but these two rectangles cannot be similar, as one of them is a square and the other is not.
(d) False. Read solution to part (c).
(e) True. Read solution to part (f).
(f) True. The area of a circle determines its radius uniquely. Hence, same area circles are congruent.

Chapter 5

5.1

5.2

5.3 For any shape, if one zooms $\times 2$ then the area will quadruple. The polygon need not be regular to solve this problem.

5.4 Since the length of and L shape is equal to a reversed L (all angles are right!), then the perimeter of the figure is equal to the perimeter of a 24 by 28 rectangle, which is $2(24+28)=104$.

5.5 Since an octagon has interior angle sum equal to $6\cdot 180^\circ$, then each interior angle measures

$$\frac{6\cdot 180^\circ}{8} = 3\cdot 45^\circ = 135^\circ$$

It follows that an exterior angle measures $180^\circ - 135^\circ = 45^\circ$.

5.6 Note that the two hexagons are similar and that the ratio of their sides is 2 : 1 (because the ratio of their apothems is 2 : 1). But that implies that the the ratio of their areas is 4 : 1. So, if the area of the small hexagon is A then the area of the big one is $4A$. It follows that the shaded area is $3A$.

Now we just need to find A, assuming a is known. The measure of every interior angle in a regular hexagon is 120°, it follows that the length of the side (say $2s$) relates to the apothem by

$$a = s\tan 60^\circ = s\sqrt{3}$$

So, the area of each of the six triangles that form the small hexagon is

$$\frac{\left(2\frac{a}{\sqrt{3}}\right)a}{2} = \frac{a^2}{\sqrt{3}} = \frac{\sqrt{3}a^2}{3}$$

and thus the area of the smaller hexagon is

$$A = 6\,\frac{\sqrt{3}a^2}{3} = 2\sqrt{3}a^2$$

Finally, the shaded area is $3A = 6\sqrt{3}a^2$.

5.7 If the polygon has n sides, then the angle sum of the polygon is $120n^\circ$. Since the angle sum of a polygon with n sides is $(n-2)180^\circ$ then

$$120n^\circ = (n-2)180^\circ$$

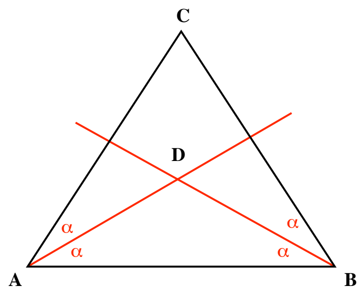
which implies $n = 6$.

5.8 This follows from the fact that each interior angle in a regular hexagon measures 120° (see previous problem), and the fact that any radius of the polygon bisects these interior angles. After that is clear, it is easy to see that the six triangles formed by the radii and the sides of a regular hexagon have congruent interior angles (60° each), and thus the triangles are equilateral.

5.9

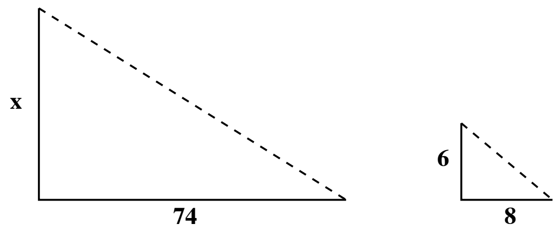
Chapter 6

6.1 We consider an isosceles triangle $\triangle ABC$ with base angles $\angle CAB$ and $\angle CBA$ with measure 2α. We draw the angle bisectors to get the following picture.



Since the base angles of $\triangle ABD$ are congruent, then this triangle is isosceles.

6.2 The picture associated to this problem is

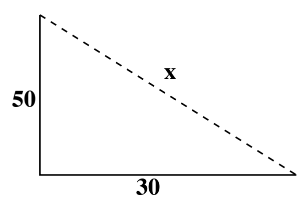


Where the dashed lines are given by solar rays. Since the angle formed by these rays are the same, and both triangles are right then the triangles are similar. It follows that

$$\frac{x}{74} = \frac{6}{8}$$

which implies $x = 55.5\ ft$

6.3 The picture associated to this problem is



Since the triangle is right then using the Pythagorean theorem yields that the dashed line (hypothenuse) is approximately $58.3\ ft$.

6.4 Since $\triangle ABE$ is inscribed on the square then both the base and the height of it are $6\ cm$. Thus the area of $\triangle ABE$ is $(6 \cdot 6)/2 = 18\ cm^2$. It follows that the area of the region external to the triangle is $6 \cdot 6 - 18 = 18\ cm^2$.

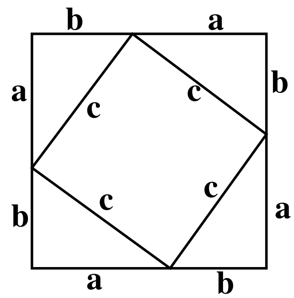
Since we know the triangle formed by all the midlines of a triangle is similar to the original triangle with ratio 0.5 then the height and base of the smaller triangle must be $3\ cm$. Hence, its area is $(3 \cdot 3)/2 = 4.5\ cm^2$.

The area of the whole purple region is $18 + 4.5 = 22.5\ cm^2$.

6.5 It is easy to see that $\angle ABE = \angle ACB = 65^\circ$ and that $\angle EAB = \angle ACD = \angle CBE = 25^\circ$. Thus $\triangle ACD \cong \triangle CAB$ by ASA. Also, $\triangle ACD \sim \triangle CAB \sim \triangle BAE \sim \triangle CBE$.

6.6

6.7 From the picture



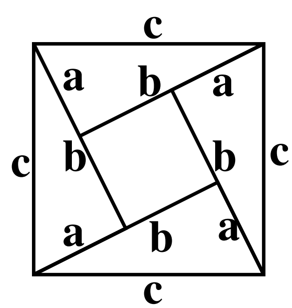
we see that the square-like figure in the middle has four congruent sides (length c) and has four right angles. Thus it is a square.

Now we compute the area in two different ways. first realizing that the figure is a square with side $a+b$, and thus the area is $(a+b)^2$. Then we compute the area by adding c^2 (the square in the middle) plus $4\,\frac{ab}{2}$ (the four triangles), we get

$$4\frac{ab}{2}+c^2=(a+b)^2$$

After some algebra we get $a^2+b^2=c^2$.

Now with the picture

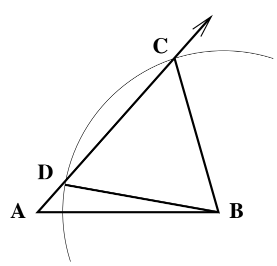


we act similarly and we see that the square in the middle has side $b-a$. Thus computing the areas yields

$$c^2=4\frac{ab}{2}+(a-b)^2$$

After some algebra we get $a^2+b^2=c^2$.

6.8 It does not, consider the following figure, where the circle passing through D and C is centered at B



Note that $\angle CAB=\angle DAB$ is an angle of both $\triangle CAB=\triangle DAB$, $\overline{AB}$ is a side of both $\triangle CAB=\triangle DAB$, and $\overline{BD}\cong\overline{BC}$ because they are radii of the same circle. Then, there is a correspondence between the vertices of $\triangle CAB$ and $\triangle DAB$ so that we get congruent Angle-Side-Side but the triangles are clearly not congruent.

6.9 No, if the side with length 7 *in* is taken as the base of the triangle, then the other two sides will never meet, as the sum of their lengths is less than 7 *in*.

6.10 This follows directly from *SSS*.

6.11

6.12

6.13

6.14

6.15

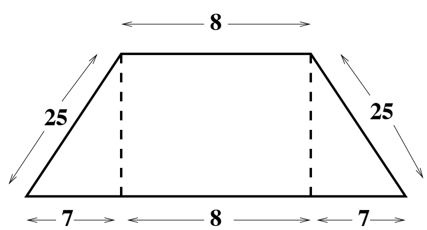
6.16

Chapter 7

7.1 Since we want to cover the field with no overhang then the width of the square must divide both 360 and 200. Now, $gcd(360,200) = 40$ then we need squares that have width at most 9 ft and that divide 40. So, the largest the squares can be is 8 $ft \times 8 ft$.

7.2 Since the diagonals in a rhombus bisect, then after tracing them we get four triangles, the sides of the rhombus are the hypothenuses of these triangles. So, using the Pythagorean theorem we get that each side is 15 *in*, and thus the perimeter is 60 *in*.

7.3 Since we know all the sides' lengths and that the trapezoid is isosceles then we get the following picture



where the dashed lines are the heights we want to measure. Note that each height can be though of as a leg of a right triangle with other leg measuring 7 *in* and hypothenuse measuring 25 *in*. It follows, by the Pythagorean theorem that $h = 24$ *in*.

7.4 Since the trapezoid is isosceles, then $\overline{RS} \cong \overline{TW}$. It follows that $\triangle SRW \cong \triangle TWR$ by *SAS*. Then, $\angle TRW \cong \angle SWR$, and thus $\triangle RPW$ is isosceles (congruent base angles).

7.5 Since $\overline{LP} \cong \overline{MP}$ then $\triangle LPM$ is isosceles, and thus $\angle PLM \cong \angle PML$. But since $\overline{AD}||\overline{LM}$ then

$$\angle LPM \cong \angle PLM \cong \angle PML \angle MPD$$

Now we use that $\overline{AP} \cong \overline{PD}$ and get that $\triangle APL \cong \triangle DPM$ by SAS. In particular, we get $\angle LAP \cong \angle MDP$, which forces $ABCD$ to be isosceles.

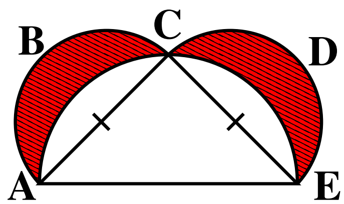
7.6 No, $\alpha > \beta$ implies that $\overline{BC}$ is longer than $\overline{AD}$.

7.7 (a) Since $\overline{AD}||\overline{BC}$ then $\angle BAE \cong \angle FCD$. Then, $\triangle BEA \cong \triangle DCF$ by SAS. It follows that $\angle CDF = \alpha$.

7.8

Chapter 8

8.1 Assume that ABC, ACE, and CDE are semicircles, and that $AC \cong CE$. We want to show that the area of the shaded region is equal to the area of $\triangle ACE$



Since the vertices of $\triangle AEC$ are on the circle, then the height of $\triangle AEC$ is r and its base is $2r$, where r is the radius of the circle through A, E and C. It follows that the area of $\triangle AEC$ is $\frac{r \cdot (2r)}{2} = r^2$ and that (using the Pythagorean theorem with the altitude of $\triangle AEC$) $r^2 + r^2 = AC^2$. So, $AC = r\sqrt{2}$.

The area of the whole figure is equal to the area of the triangle plus the area of the semicircles, we get

$$r^2 + \pi \left(\frac{r\sqrt{2}}{2} \right)^2 = r^2 \left(1 + \frac{\pi}{2} \right)$$

Since the shaded area is obtained from subtracting the white semi-circle from the whole thing, we get that the area of the shaded area is

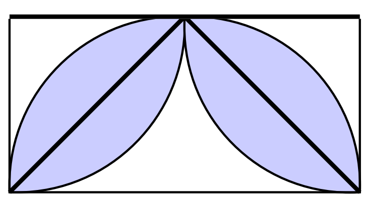
$$r^2 \left(1 + \frac{\pi}{2} \right) - \frac{1}{2}\pi r^2 = r^2$$

So, the area of the triangle equals the area of the shaded region.

8.2 Note that the area of the red region is the area of the circle minus the area of the white square (the blue parts are irrelevant at this point). So let r be the radius of the big circle. It follows that the diagonal of the square is $2r$, which implies that the side of the square is $a = r\sqrt{2}$. So, the red region has area

$$\pi r^2 - (r\sqrt{2})^2 = r^2(\pi - 2)$$

The are of the blue region is found by looking at the bottom part, which is



The area of the rectangle is is r^2 because it is half of the area of the square, the area of the semicircle is

$$\frac{1}{2}\pi \left(\frac{r\sqrt{2}}{2} \right)^2 = \frac{\pi r^2}{4}$$

because it is half of the area of a circle with radius $\frac{r\sqrt{2}}{2}$. Finally the area of the triangle in the figure is half of the area of the rectangle. So, the area of the two top pieces (above the triangle) in blue is

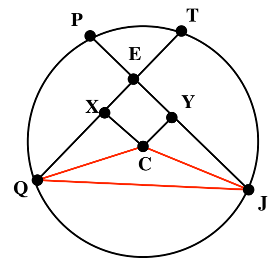
$$\frac{\pi r^2}{4} - \frac{r^2}{2} = \frac{r^2}{4}(\pi - 2)$$

Since the blue figure has 8 parts like the TWO we have found the area of then the blue area is

$$4\frac{r^2}{4}(\pi-2)=r^2(\pi-2)$$

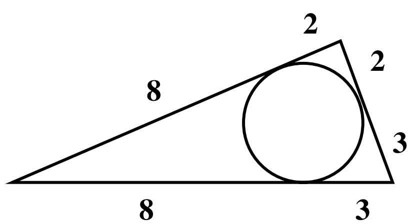
8.3

8.4 Assuming that $CXEY$ is a square, we want to show that the arc QP is congruent to the arc JT.
First join the center with Q and J to get $\triangle QCJ$



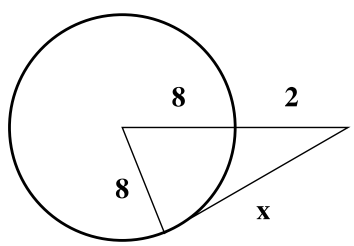
Note that $\triangle CYJ \cong \triangle CXQ$ because both are right triangles, $\overline{QC} \cong \overline{JC}$, and $\overline{CX} \cong \overline{CY}$. This implies that $\overline{QX} \cong \overline{JY}$. Since $\overline{XE} \cong \overline{EY}$ then $\overline{QE} \cong \overline{JE}$. It follows that $\triangle QJE$ is isosceles with base $\overline{QJ}$, and thus $\angle QJP \cong \angle TQJ$, which implies that arc QP is congruent to arc JT.

8.5 Since the sides of the triangle are tangent to the circle then two consecutive sides can be considered as tangents coming out of the same point (the vertex). It follows that these two segments from the vertex to the circle are congruent. Hence,



and thus $x = 11$.

8.6 The picture for this is something like



Since a tangent is always perpendicular to the radius at the tangent point, then we can use the Pythagorean theorem

$$8^2+x^2=10^2$$

which implies $x = 6$.

8.7 The square has perimeter 36 *in* and thus its side has length 6 *in*. Since the circle is circumscribed about the square then the diameter of the circle is equal to the length a diagonal of the square. Using the Pythagorean theorem with two sides of the square and the diagonal we get $6^2+6^2=d^2$, and thus $d=6\sqrt{2}$. It follows that the circumference of the circle is $6\sqrt{2}\pi$ *in*.

8.8

Chapter 9

9.1 Let the edges of the box have length x, $2x$ and $3x$. Then the main diagonal in the box has length

$$\sqrt{x^2+(2x)^2+(3x)^2}=7\sqrt{2}$$

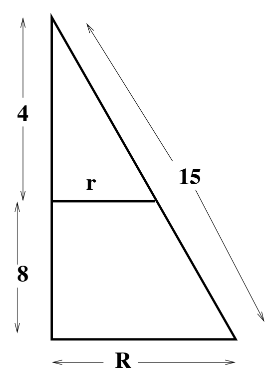
It follows that $14x^2=98$, and thus $x=\sqrt{7}$. The volume of the box is

$$\sqrt{7}\cdot(2\sqrt{7})\cdot(3\sqrt{7})=42\sqrt{7}$$

9.2 We cut the cylinder into two smaller cylinders, each one containing a full cone (half of the sand-clock). We know that the volume of a cone inscribed in this way in a cylinder is a third of the volume of the cylinder. It follows that the total volume of the cylinder is the sum of the volumes of the cylinders, and thus the volume of the sand-clock is the sum of the volumes of the cones. Since each cone's volume is a third of the smaller cylinders', then the volume of the sand-clock is a third of the volume of the large cylinder.

9.3 We need to find the surface area of the can.
The top and bottom have area $\pi(2^2)=4\pi\ cm^2$ each.
The side of the cylinder has area $2\pi(4)(7)=56\pi\ cm^2$.
It follows that the can has total surface area $64\pi\ cm^2$.

9.4 Drop the altitude from the tip of the larger cone to the center of the base (this line also goes through the small cone's center). Note that the slant(line), the altitude and the radii of the bases form the following figure.



where R is the radius of the large cone's base and r the radius of the small cone's base.

Using the Pythagorean theorem we find that $R=9$, and then using similarity of triangles we get that $r=3$ and that the slant of the smaller cone is 5.

Since the larger cone has been obtained from the smaller one by zooming $\times 3$ then the volume of the larger cone is 27 times the volume of the smaller cone.

9.5 This is like a $3-D$ Pythagorean theorem. Take a face of the cube. We know the diagonal of that cube is $y\sqrt{2}$ by using the Pythagorean theorem. Now the diagonal in the cube can be thought of as the hypothenuse of a triangle with legs a 'vertical' side of the cube and the diagonal of the square at the base of the cube. Thus, the Pythagorean theorem says.

$$x^2=y^2+(y\sqrt{2})^2$$

which implies $x=y\sqrt{3}\ in$.

Another way to do this is to use coordinates. We set one corner of the cube at $(0,0,0)$ and the other must be at (y,y,y). So, the distance between these points is

$$d=\sqrt{(y-0)^2+(y-0)^2+(y-0)^2}=\sqrt{3y^2}=y\sqrt{3}$$

9.6 Since the radius is a zoomed $\times 3$ then all lengths are multiplied by 3, and thus all areas will be multiplied by 3^2 and all volumes by $3^3 = 27$.

9.7 The volume of the box is $3 \cdot 8 \cdot 9 = 216\ in^3$. Since $6^3 = 216$, then the side of the cube is 6 inches, and thus its side has area $36\ in^2$.

9.8

9.9

Chapter 10

10.1 We can see that the coordinates of the points are $A = (-2,2)$ and $B = (3,-1)$. It follows that the slope of the line that joins them is

$$\frac{-1-2}{3-(-2)} = -\frac{3}{5}$$

The equation of the line through A and B can be found by using that A is a point of

$$y = -\frac{3}{5}x + b$$

We get $b = \frac{4}{5}$, and thus the line has equation

$$y = -\frac{3}{5}x + \frac{4}{5}$$

A line perpendicular to the line above must look like

$$y = \frac{5}{3}x + b$$

Since $(1,2)$ is on this line then $b = \frac{1}{3}$ and thus

$$y = \frac{5}{3}x + \frac{1}{3}$$

The intersection of these two lines is found by solving the system of linear equations

$$y = -\frac{3}{5}x + \frac{4}{5} \qquad\qquad y = \frac{5}{3}x + \frac{1}{3}$$

which we re-write

$$5y = -3x + 4 \qquad\qquad 3y = 5x + 1$$

This system has solution $\left(\frac{7}{34}, \frac{23}{34}\right)$.

10.2

10.3

10.4 Set a vertex of the rectangle to be the origin of the plane, and two of its sides to lie on the x-axis and y-axis. It follows that the four vertices of the rectangle are $(0,0), (a,0), (a,b)$, and $(b,0)$.

The slopes of the lines that 'contain' the diagonals of the rectangle are:

$$m_{(0,0)-(a,b)} = \frac{b}{a} \qquad\qquad m_{(a,0)-(0,b)} = -\frac{b}{a}$$

Since we are assuming the diagonal are perpendicular then the product of these slopes must be equal to -1. We get,

$$a^2 = b^2$$

and thus $a = \pm b$. But since both a and b are positive, then $a = b$ and thus the rectangle is a square.

10.5

10.6 With the same setting as the solution to problem 10.4. We first find the equations of the lines that 'contain' the two diagonals. We already know the slopes, and also their y-intercepts, so it is easy to see that the lines are

$$y = \frac{b}{a}x \qquad\qquad y = -\frac{b}{a}x + b$$

These lines intersect at $\left(\frac{a}{2}, \frac{b}{2}\right)$, which is the midpoint of both of the diagonals.

10.7

10.8 (a) The slope of the line is

$$m = \frac{(a-1)-(a+1)}{(3a+5)-a} = \frac{-2}{2a+5}$$

Now using that $A = (a, a+1)$ is on the line we get

$$a+1 = \frac{-2}{2a+5}(a) + b$$

which forces $b = \frac{2a^2+9a+5}{2a+5}$. And thus, the equation of the line is

$$y = \frac{-2}{2a+5}x + \frac{2a^2+9a+5}{2a+5}$$

(b) Any line perpendicular to the line above must have slope $\frac{2a+5}{2}$. So, if the line goes through $A = (a, a+1)$ then

$$a+1 = \frac{2a+5}{2}(a) + b$$

which implies $b = \frac{2-2a^2-3a}{2}$. And thus, the line asked for is

$$y = \frac{2a+5}{2}x + \frac{2-2a^2-3a}{2}$$

(c) As mentioned above, this line must have slope $\frac{2a+5}{2}$. So, if the line goes through $A = (3a+5, a-1)$ then

$$a-1 = \frac{2a+5}{2}(3a+5) + b$$

which implies $b = \frac{-6a^2-23a-27}{2}$. And thus, the line asked for is

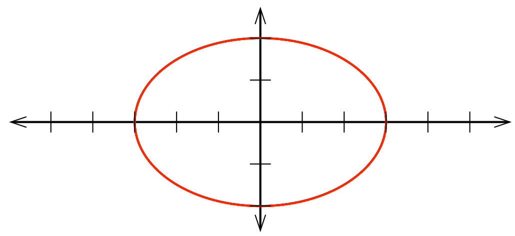
$$y = \frac{2a+5}{2}x + \frac{-6a^2-23a-27}{2}$$

Chapter 11

11.1 We are given $4x^2+9y^2=36$. We divide by 36 to get

$$\frac{x^2}{9}+\frac{y^2}{4}=1$$

It follows that the ellipse is centered at $(0,0)$ and $a=3$ and $b=2$. A picture follows



Finally, since $a^2=c^2+b^2$, then $c=\sqrt{5}$, and thus the foci are at $(\pm\sqrt{5},0)$.

11.2 This is the same as above, because we are just interchanging x and y. So, the ellipse is 'vertical' centered at $(0,0)$, and the foci are $(0,\pm\sqrt{5})$.

11.3

11.4 It is clear that the center of the ellipse is $(1,-1)$ and that $a=\sqrt{5}$ and $b=\sqrt{3}$. Since $a^2=c^2+b^2$, then $c=\sqrt{2}$. It follows that the foci are at $(1\pm\sqrt{2},-1)$.

11.5 We find the center of the circle by finding the midpoint of the diameter given.

$$\left(\frac{1+5}{2},\frac{1+4}{2}\right)=(3,2.5)$$

Since the distance between $(1,1)$ and $(5,4)$ is

$$\sqrt{(5-1)^2+(4-1)^2}=\sqrt{4^2+3^2}=5$$

then the radius of the circle is 2.5. It follows that the equation of the circle is

$$(x-3)^2+(y-2.5)^2=(2.5)^2$$

11.6

11.7 $x-h=4p(y-k)^2$

11.8 We find the center of the circle by finding the midpoint of the diameter given.

$$\left(\frac{3+6}{2},\frac{1+5}{2}\right)=(4.5,3)$$

Since the distance between $(3,1)$ and $(6,5)$ is

$$\sqrt{(6-3)^2+(5-1)^2}=\sqrt{3^2+4^2}=5$$

then the radius of the circle is 2.5. It follows that the equation of the circle is

$$(x-4.5)^2+(y-3)^2=(2.5)^2$$

11.9

Chapter 12

12.1 We notice that the translation maps $(1,2)$ to $(-2,6)$, thus the vector used for the translation is $(-2,6)-(1,2)=(-2-1,6-2)=(-3,4)$. Note that we could have used any of the four corners of the rhombus to obtain this vector.

12.2 Since a reflection R across the x-axis maps a point (x,y) to $(x,-y)$ then

$$R(-3,2)=(-3-2) \qquad R(-1,4)=R(-1,-4) \qquad R(-2,6)=(-2,-6)$$

It follows that $A'=(-3-2)$, $B'=R(-1,-4)$, and $C'=(-2,-6)$.

12.3

12.4

Chapter 13

13.1 Since the number of possible outcomes is 20, and there are 8 prime numbers less than 21 $(2,3,5,7,11,13,17,$ and 19) then the probability is

$$P(X)=\frac{8}{20}=\frac{2}{5}=0.4$$

which is a 40% chance.

13.2 The total number of markers in my bag is 43, the total number of red markers is 12, thus

$$P(X)=\frac{12}{43}\sim .28$$

which is a 28% chance.

13.3 The area of the square is 100, and thus this will be considered to be the 'total number of outcomes'. The diameter of the circle equals the length of the side of the square, and thus its area is $5^2\pi=25\pi$. Hence, the probability asked for is

$$P(X)=\frac{25\pi}{100}=\frac{\pi}{4}\sim 0.78$$

which is about a 78% chance.

13.4 Since months have not all the same number of days then in order to compute this we need to look at days, not at months.

In this problem, the possible outcomes are the days of the year (no leap year), so there are 365 of them. The favorable outcomes are the days of the months ending in "ary". These months are January (31 days) and February (28 days). So, there are 59 favorable outcomes. It follows that

$$P(X)=\frac{59}{365}\sim 0.16$$

which is approximately 16%.

13.5

Chapter 14

14.1

14.2 If the data is distributed in a normal curve (bell-shaped) then the highest point of the curve is exactly at the middle of the data values, that means that it is exactly where the median is. Moreover, since that is the highest point then the frequency of the median is as large as possible, and thus the median is also the mode. Finally, since the data is symmetrically distributed then whenever there is a value that is, let us say, 1 unit less than the median, then there is another that balances it out because it must be 1 unit larger than the median. This means that the mean is also equal to the median.

14.3

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