This has been taken out of the lecture notes for another class, it is a much more detailed analyisis of the postulates, with more proofs and comments.
Do not worry about this Neutral Geometry stuff, it is not quite relevant for us.

Chapter 4

Basic neutral geometry

Arguably the most important math book in history is Euclid's Elements. A 13 volume treaty, written about 2300 years ago, that compiled all the mathematical knowledge of that time in a increasing manner, meaning that at the beginning one can find the very basic and essential and then later it all gets more complex and fancier. So, the first book contains definitions and a few 'obvious' or 'evident' facts that needed no proof and that will be crucial to develop future results.

These evident facts were called common notions and postulates. The earlier are general ideas, not necessarily geometric. The latter are purely geometric and contain a very controversial 'evident' fact... *the fifth*. Most of the theorems that are taught in schools today are in the Elements. Euclidean has shown to be great to solve 'real life' problems, as is what engineers use to design buildings, bridges, etc.

The study of shapes and their properties (sometimes called geometry) that avoids as much as possible the use of *the fifth* is called neutral geometry. In order to learn about this geometry we need to review the work of Euclid first.

Euclid's Elements

What follows has been taken out of Book *I* of Euclid's Elements.

Visit

$$http://aleph0.clarku.edu/\sim djoyce/java/elements/elements.html$$

for a more complete review of this material.

Definitions

1. A point is that which has no part. **We would say now 'it has no dimensions'**
2. A line is breadthless length. **We would say now 'it is one-dimensional'**
3. The ends of a line are points. **Thinking mostly about segments, what are the ends of a line?**
4. A straight line is a line which lies evenly with the points on itself. **Kind of using the idea of a line being self-symmetric**
5. A surface is that which has length and breadth only. **We would say now 'it has two dimensions'**
6. The edges of a surface are lines. **Similar thing as the ends of lines**
7. A plane surface is a surface which lies evenly with the straight lines on itself. **A plane must contain all the lines through any of its points**

8. A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.
9. And when the lines containing the angle are straight, the angle is called rectilinear.
10. When a straight line standing on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, and the straight line standing on the other is called a perpendicular to that on which it stands.
11. An obtuse angle is an angle greater than a right angle.
12. An acute angle is an angle less than a right angle.
13. A boundary is that which is an extremity of anything.
14. A figure is that which is contained by any boundary or boundaries.
15. A circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure equal one another.
16. And the point is called the center of the circle.
17. A diameter of the circle is any straight line drawn through the center and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle.
18. A semicircle is the figure contained by the diameter and the circumference cut off by it. And the center of the semicircle is the same as that of the circle.
19. Rectilinear figures are those which are contained by straight lines, trilateral figures being those contained by three, quadrilateral those contained by four, and multilateral those contained by more than four straight lines. **Polygons!**
20. Of trilateral figures, an equilateral triangle is that which has its three sides equal, an isosceles triangle that which has two of its sides alone equal, and a scalene triangle that which has its three sides unequal.
21. Further, of trilateral figures, a right-angled triangle is that which has a right angle, an obtuse-angled triangle that which has an obtuse angle, and an acute-angled triangle that which has its three angles acute.
22. Of quadrilateral figures, a square is that which is both equilateral and right-angled; an oblong that which is right-angled but not equilateral; a rhombus that which is equilateral but not right-angled; and a rhomboid that which has its opposite sides and angles equal to one another but is neither equilateral nor right-angled. And let quadrilaterals other than these be called trapezia.
23. Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction. **... being in the same plane... Important!!! When lines do not intersect in the $3-D$ space, it is common to say they are 'skew'**

Common Notions

1. Things which equal the same thing also equal one another. **Transitivity**
2. If equals are added to equals, then the wholes are equal.
3. If equals are subtracted from equals, then the remainders are equal.
4. Things which coincide with one another equal one another.
5. The whole is greater than the part.

Postulates

Let the following be postulated: **Important: kind of assumed here is that all elements considered are on the same plane, also assumed that lines have infinite length**

1. To draw a straight line from any point to any point. **A unique line!**

2. To produce a finite straight line continuously in a straight line.

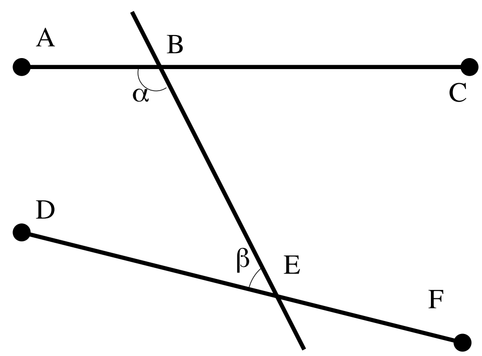
3. To describe a circle with any center and radius. **Note that the center and the radius must be known**

4. That all right angles equal one another.

5. That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

About the 5^{th}:

(a) In a picture, the fifth says that if



then since the sum of the angles α and β is less than 180^o, then the lines AC and BF, once extended forever, should intersect 'on the side' of the line BE where α and β are.

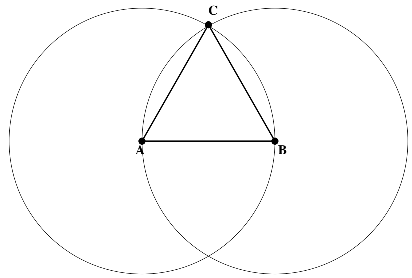
(b) What does 'on that side' mean? We are assuming some kind of order or orientation, we never had this in chapter 2.

(c) Euclid does not use this postulate until proposition 29, thus the first 28 are valid in neutral geometry.

The first 28 propositions

1. To construct an equilateral triangle on a given finite straight line.

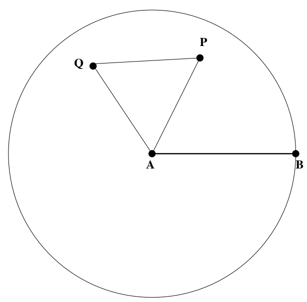
   Proof. Given segment $\overline{AB}$ with length r. First draw circles with centers A and B and radius r (postulate 3). The intersection of these circles is called C (note we have two options for C, choose either). The triangle $\triangle ABC$ is equilateral with the given base (we use postulate 1 to construct the triangle).



∎

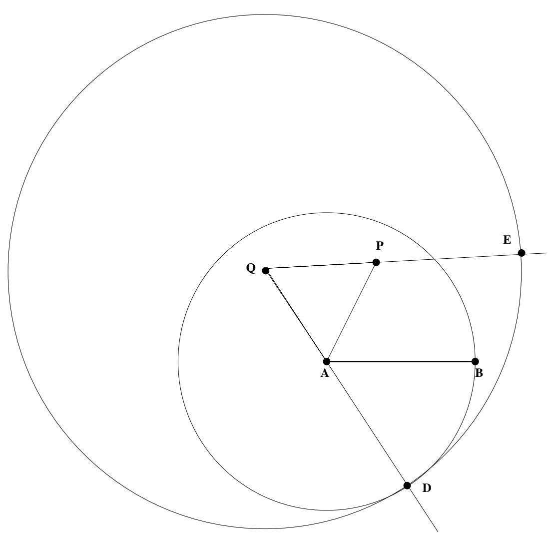
2. To place a straight line equal to a given straight line with one end at a given point.

   Proof. Given segment $\overline{AB}$ and a point P. First join P and A with a line (postulate 1). Using proposition 1 we can find Q such that $\triangle APQ$ is equilateral. Now draw a circle centered at A with B on the boundary (postulate 3).



   Extend the segments $\overline{QA}$ andn $\overline{QP}$ into lines (postulate 2), call D and E to the intersections of these lines and the circle already constructed. Now draw a circle centered at Q with D on the boundary (postulate 3).

   Note that $\overline{AB} \cong \overline{AD}$ and that $\overline{QD} \cong \overline{QE}$. Since $\overline{QA} \cong \overline{QP}$, it follows that $\overline{AB} \cong \overline{AD} \cong \overline{PE}$. ∎

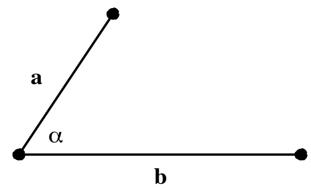


3. To cut off from the greater of two given unequal straight lines a straight line equal to the less.

   Proof. Using the previous proposition we copy the shorter segment and put it at one extreme of the longer segment. It follows that we can now cut off (literally) the shorter segment from the longer. ∎

4. If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides. **Better known as SAS**

   Proof. Assume that two lengths, a and b, are given and the angle α between them, thus we have a picture like.

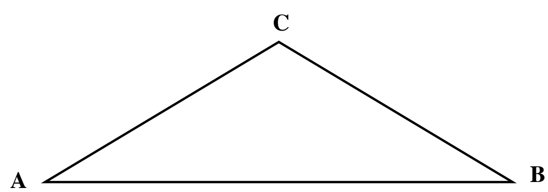
   

   It is pretty clear that there is a unique way to complete that picture to create a triangle. Done. ∎

5. In isosceles triangles the angles at the base equal one another, and, if the equal straight lines are produced further, then the angles under the base equal one another.

6. If in a triangle two angles equal one another, then the sides opposite the equal angles also equal one another.

   Proof. (Prop. 5 and 6.) We will represent an isosceles triangle as

   where $\overline{AC} \cong \overline{BC}$.

   Extend $\overline{AC}$ and $\overline{BC}$ downwards (using the picture above) the same distance (using proposition 2) to obtain a point F on $\overleftrightarrow{AC}$ and a point G on $\overleftrightarrow{BC}$, thus creating two triangles $\triangle FAC$ and $\triangle GAB$. Since $\overline{FA} \cong \overline{GA}$, $\angle ACB$ is common to both triangles, and $\overline{AC} \cong \overline{BC}$ then $\triangle FAC \cong \triangle GAB$ (by SAS). It follows that $\angle FBC \cong \angle GCB$ thus forcing the base angles of $\triangle ABC$ to be congruent.

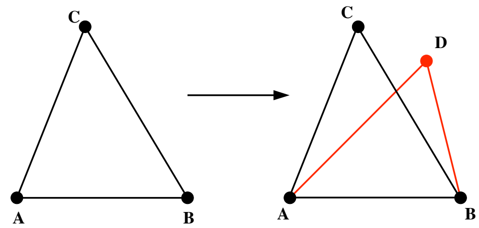
   Moreover, if $\triangle ABC$ above has congruent base angles (but we don't know yet it is isosceles), then we could assume that $\overline{AC}$ is larger than $\overline{BC}$. Using proposition 3 we can find a point D on $\overline{AC}$ such that $\overline{AD} \cong \overline{BC}$. Now consider $\triangle ABC$ and $\triangle BAD$, note that $\angle ABC \cong \angle DAB$, $\overline{BC} \cong \overline{AD}$, and that $\overline{AB}$ is common to both triangles, thus $\triangle ABC \cong \triangle BAD$ (by SAS).

   It follows that the area of $\triangle ABC$ equals the area of $\triangle BAD$, which is a contradiction. Hence, using a similar argument for when $\overline{BC}$ is larger than $\overline{AC}$, we get that $\overline{AC} \cong \overline{BC}$, and thus the triangle must be isosceles. ∎

   Note that there is another proof for proposition 5 that uses the angle bisector from C, and a proof for 6 that uses ASA. We cannot write those proofs at this points because we haven't proved ASA and we haven't learned how to construct the angle bisector of a given angle.

7. Given two straight lines constructed from the ends of a straight line and meeting in a point, there cannot be constructed from the ends of the same straight line, and on the same side of it, two other straight lines meeting in another point and equal to the former two respectively, namely each equal to that from the same end.

   Proof. This confusing proposition says that if $\triangle ABC$ is given then it is impossible to construct a second triangle $\triangle ABD$, like in the picture below, such that $\overline{AC} \cong \overline{AD}$ and $\overline{BC} \cong \overline{BD}$.



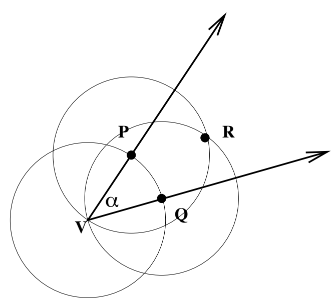
If we assume such a triangle can be constructed then we join C and D and consider $\triangle ACD$ and $\triangle BCD$, both isosceles by assumption. It follows that $\angle ACD \cong \angle ADC$ and that $\angle DCB \cong \angle CDB$.

Since $\angle DCB$ is 'contained' in $\angle ACD$ then $\angle DCB < \angle ADC$ (measures, that is), an similarly $\angle CDB > \angle ADC$. So, $\angle CDB > \angle DCB \cong \angle CDB$, which is impossible. ■

8. If two triangles have the two sides equal to two sides respectively, and also have the base equal to the base, then they also have the angles equal which are contained by the equal straight lines. **Better known as SSS**

9. To bisect a given rectilinear angle.

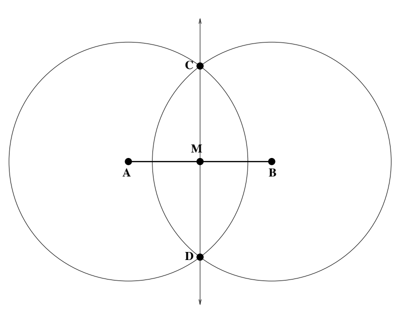
   Proof. We first draw a circle centered at the vertex V of α. The intersections of this circle with the sides of the angle are labeled P and Q. Now we draw two circles **with the same radius** centered at P and Q. These circles intersect at R (note we have two choices for R, choose either).



   Note that the triangles $\triangle VPR$ and $\triangle VQR$ are congruent by SSS. It follows that $\angle PVR \cong \angle QVR$. Thus $\overrightarrow{VR}$ is the bisector of α. ■

10. To bisect a given finite straight line.

   Proof. The segment $\overline{AB}$ is given. We draw circles **with the same radius** centered at A and B. These circles intersect in two points, which we label C and D. The line through C and D intersects $\overline{AB}$ at a point M. We claim that M is the midpoint of $\overline{AB}$. Moreover, that the line $\overleftrightarrow{CD}$ is the perpendicular bisector of $\overline{AB}$.



   The claim follows from the fact that the points C and D are equidistant from A and B and thus $\triangle CBD \cong \triangle CAD$. It follows that $\angle BCD \cong \triangle ACD$, which implies that $\triangle CMB \cong \triangle CMA$ by SAS. Hence $\overline{AM} \cong \overline{MB}$.

The fact that $\overleftrightarrow{CD}$ is the perpendicular bisector of $\overline{AB}$ follows from the fact that the angles $\angle AMC$ is congruent to $\angle CMB$ and that their sum is 180^o. ■

11. To draw a straight line at right angles to a given straight line from a given point on it.

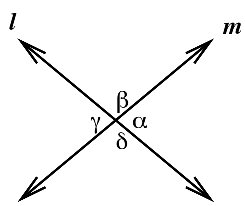
12. To draw a straight line perpendicular to a given infinite straight line from a given point not on it. **The previous two propositions allow us to draw all types of perpendiculars to a given line**

13. If a straight line stands on a straight line, then it makes either two right angles or angles whose sum equals two right angles.

14. If with any straight line, and at a point on it, two straight lines not lying on the same side make the sum of the adjacent angles equal to two right angles, then the two straight lines are in a straight line with one another.

15. If two straight lines cut one another, then they make the vertical angles equal to one another. Corollary. If two straight lines cut one another, then they will make the angles at the point of section equal to four right angles. **VAT**

    Proof.

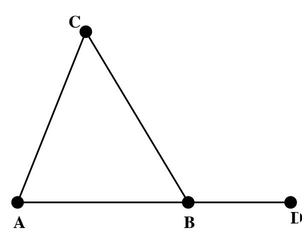


    Since $\alpha + \beta = 180^o$ and $\beta + \gamma = 180^o$, then $\alpha = \gamma$.

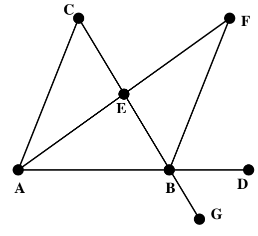
    Similarly, we get that $\beta = \delta$. ■

16. In any triangle, if one of the sides is produced, then the exterior angle is greater than either of the interior and opposite angles.

    Proof. We want to show that in the picture below $\angle CBD$ is larger than both $\angle CAB$ and $\angle ACB$.



    Using proposition 10 we find the midpoint of $\overline{CB}$, call it E. Then we draw the line from A to E, and using proposition 2, and postulate 2, we find a point F on it such that $\overline{AE} \cong \overline{EF}$. We obtain the picture



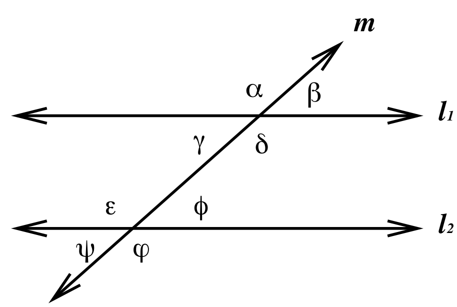
where E is the midpoint of both $\overline{CB}$ and $\overline{AF}$. It follows that $\triangle AEC \cong \triangle FEB$ by SAS, and thus $\angle ACB \cong \angle FBE < \angle EBD$.

We obtain that $\angle CAB < \angle EBD$ in a similar way. ■

Note that in proof above, we need to find the point F by extending a line until it reaches a desired length. This is done by assuming that lines extend indefinitely. It follows that for this proposition to work we need infinite lines.

17. In any triangle the sum of any two angles is less than two right angles.
18. In any triangle the angle opposite the greater side is greater.
19. In any triangle the side opposite the greater angle is greater.
20. In any triangle the sum of any two sides is greater than the remaining one. **This is the triangle inequality**
21. If from the ends of one of the sides of a triangle two straight lines are constructed meeting within the triangle, then the sum of the straight lines so constructed is less than the sum of the remaining two sides of the triangle, but the constructed straight lines contain a greater angle than the angle contained by the remaining two sides.
22. To construct a triangle out of three straight lines which equal three given straight lines: thus it is necessary that the sum of any two of the straight lines should be greater than the remaining one. **Kind of a non-constructability theorem**
23. To construct a rectilinear angle equal to a given rectilinear angle on a given straight line and at a point on it. **We can 'copy' angles**
24. If two triangles have two sides equal to two sides respectively, but have one of the angles contained by the equal straight lines greater than the other, then they also have the base greater than the base.
25. If two triangles have two sides equal to two sides respectively, but have the base greater than the base, then they also have the one of the angles contained by the equal straight lines greater than the other.
26. If two triangles have two angles equal to two angles respectively, and one side equal to one side, namely, either the side adjoining the equal angles, or that opposite one of the equal angles, then the remaining sides equal the remaining sides and the remaining angle equals the remaining angle. **Better known as ASA**
27. If a straight line falling on two straight lines makes the alternate angles equal to one another, then the straight lines are parallel to one another.
28. If a straight line falling on two straight lines makes the exterior angle equal to the interior and opposite angle on the same side, or the sum of the interior angles on the same side equal to two right angles, then the straight lines are parallel to one another.

    The previous two propositions say that, in the following picture, if $\gamma = \psi$ or $\gamma = \phi$, then l_1 is parellel to l_2



Since by now we have covered all what Euclid was able to show without using *the fifth* we will now abandon Euclid's work and we will study the work of more recent mathematicians.

What follows is a disgression into statements that are equivalent to Postulate 5.

Equivalent to the fifth

One thing we must be aware of is the facts that many geometric properties we take as granted are not that obvious as we think. In fact, when studying neutral geometry we can realize that there are impossible to prove. Many times these results are proved using the fifth, and thus they need to be generalized to be true when not considering that postulate.

It is possible to show that the fifth is actually equivalent to certain statements (by working in the realm of neutral geometry). We will discuss a few of these next.

Theorem 18 *The following statements are all equivalent*

1. *If two lines are cut by a transversal so that the angles formed on one side of the transversal add up to less than* 180^o *then the lines meet on that side of the line (AKA, the fifth)*
2. *By a point* P *exterior to a line* ℓ *passes exactly one line parallel to* ℓ *(AKA Playfair's axiom).*
3. *The angle sum of any triangle is equal to two right angles. In other words, each triangle has defect zero.*
4. *The summit angles in a Saccheri quadrilateral are right.*
5. *The Pythagorean theorem.*
6. *In any triangle, each exterior angle equals the sum of the two remote interior angles.*
7. *If two parallel lines are cut by a transversal, the alternate interior angles are equal, and the corresponding angles are equal. (AKA proposition* 29*).*
8. *There exists some triangle with angle sum is two right angles.*
9. *There exists an isosceles right triangle with angle sum equal to two right angles.*

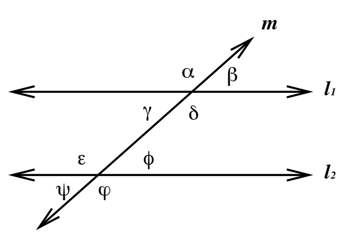
10. *There exists arbitrarily large isosceles right triangles with angle sum equal to two right angles.*
11. *The angle sum of any pair of triangles is always the same.*
12. *There exists a pair of similar, but not congruent, triangles.*
13. *Every triangle can be circumscribed.*
14. *The fourth angle in a Lambert quadrilateral is right.*
15. *Rectangles exist.*
16. *There exists a pair of straight lines that are at constant distance from each other.*
17. *Two lines that are parallel to the same line are also parallel to each other.*
18. *Given two parallel lines, any line that intersects one of them also intersects the other.*
19. *There is no upper limit to the area of a triangle.*
20. *The area of a triangle is half its base times its height.*
21. *There exists a circle passing through any three noncollinear points.*
22. *The circumference of any circle of radius r is $2\pi r$.*
23. *The area of any circle of radius r is πr^2.*
24. *For any triangle, there exists a similar noncongruent triangle.*
25. *Opposite sides of a parallelogram are congruent.*

Let us first consider proposition 29, which can be considered as the converse of propositions 27 and 28.

Proposition 29 A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the sum of the interior angles on the same side equal to two right angles.

Now we will show that it is equivalent to the fifth.

Proof. Let us steal the picture from propositions 27 and 28 for this proof.



First assume the fifth, and that in the picture above $l_1 || l_2$ but, for example, $\beta \neq \phi$.

If $\beta - \phi < 0$ then $\gamma + \epsilon = \beta + (\pi - \phi) = \pi + (\beta - \phi) < \pi$, thus by the fifth the lines must intersect to the left of m. Contradiction.

If $\beta - \phi > 0$ then $\delta + \phi = (\pi - \beta) + \phi = \pi - (\beta - \phi) < \pi$, thus by the fifth the lines must intersect to the right of m. Contradiction.

Hence $\beta = \phi$.

Now assume proposition 29 and let us prove the fifth. So, in the picture above we assume that $\delta + \phi < \pi$.

If $l_1 || l_2$ then proposition 29 implies that $\beta = \phi$, and thus $\delta + \phi = \delta + \beta = \pi$. Contradiction.

If the lines intersect to the left of m, then we have a triangle with vertices $l_1 \cap l_2$ and $l_1 \cap m$ and $l_2 \cap m$. Two of this triangle's interior angles are $\gamma = \pi - \delta$ and $\epsilon = \pi - \phi$. Since $\delta + \phi < \pi$ then the sum of these two angles is

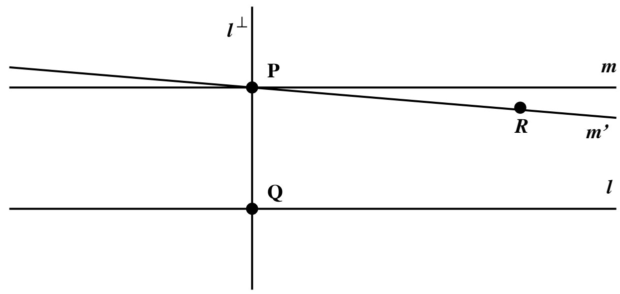
$$(\pi - \delta) + (\pi - \phi) = 2\pi - (\delta + \phi) > \pi$$

which is a contradiction. ■

The next proof shows that the fifth is equivalent to Playfair's axiom.

Proof. Let us assume the fifth. Let l be a line and P a point not on l. We construct the unique line $l^{\perp}$ that is perpendicular to l through P. Let $Q = l^{\perp} \cap l$. Now we construct the unique perpendicular m to $l^{\perp}$ through P. It follows that $l^{\perp}$ is a common perpendicular to m and l, and thus $m||l$.

Now take any other line m' through P. We get the next picture



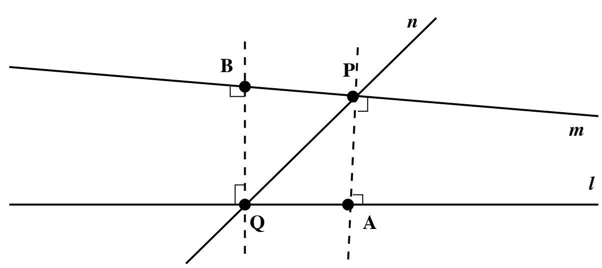
Following the picture, note that the $\angle QPR$ is less than 90° and thus the sum of the angles formed to the right of $l^{\perp}$ by this line and l and m' add up to less than 180°. By the fifth l should intersect m' to the right of $l^{\perp}$.

Note that depending on the picture we might need to 'relocate' the point R so we can repeat the argument above. Since this is always possible, then m is the only parallel to l through P.

Now let us assume Playfair's axiom. Let us consider two lines l and m and a transversal n such that the angles to (let us say) the right of n formed by n, and l and m add up to less than 180°, also assume that $l||m$.

Note that if n is perpendicular to l then Playfair's axiom and the construction of a parallel used in the previous part of this proof force $n \perp m$.

If n is not perpendicular to l, then using Playfair's axiom and the previous construction we get that n is not perpendicular to m as well. Now drop the perpendiculars to l through P and to m through Q. Because if the construction used before and the uniqueness of parallels (Playfair's axiom) then these lines are common perpendiculars to l and m. We get,



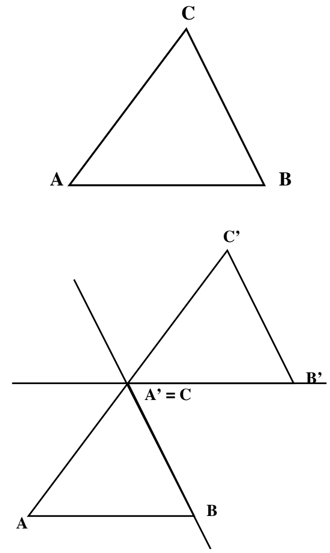
It follows that $QAPB$ is a rectangle! And thus by theorem 13 $\overline{QA} \cong \overline{BP}$ and $\overline{QB} \cong \overline{AP}$. It follows that $\triangle QPB \cong \triangle PQA$ by SSS. But this forces that $\angle PQA + \angle QPA = 90°$, which contradicts our assumption of having angles to the right of n with sum less than 180° (or that n is not perpendicular to l). ■

Next we show that Playfair's axiom / fifth postulate is equivalent to the angle sum of a triangle is two right angles (180°, when among friends).

Proof. Let us assume the fifth, and consider a triangle $\triangle ABC$

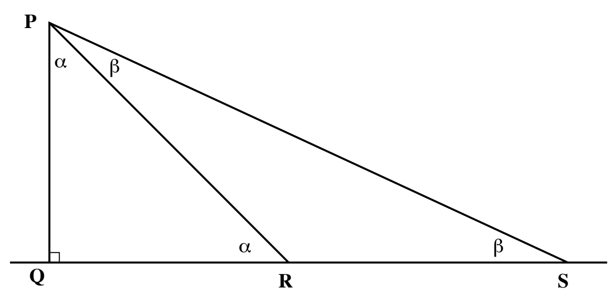
We draw a parallel line to $\overleftrightarrow{AB}$ through C and then we extend the sides of $\triangle ABC$, and then we create a copy of $\triangle ABC$ as in the picture below. Note that to create that figure above we are using proposition 29 (proved using the fifth).

Now, since $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ are parallel then the angle sum of $\triangle ABC$ is equal to the sum of the angles at $A' = C$ that are above $\overleftrightarrow{A'B'}$, which is 180^o.



Now we want to show the fifth/Playfair assuming that the angle sum of every triangle is two right angles. But first we need to show a little lemma assuming the angle-sum of any triangle is 180°. Let P be a point not on l, and θ any given angle, then it is possible to find a point $T \in l$ such that the angle formed by $\overleftrightarrow{PT}$ and l is less than θ.

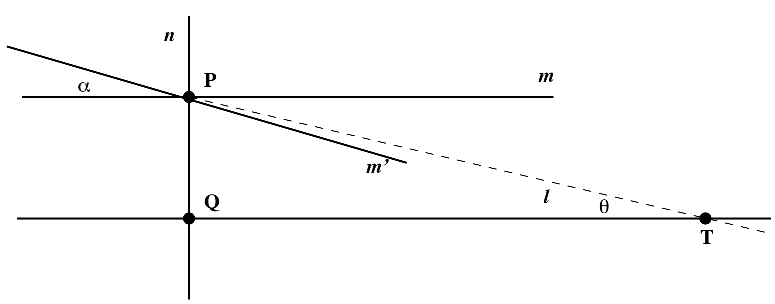
The construction starts by dropping a perpendicular to l through P, let Q be the point of intersection of l and this perpendicular. Now let $R \in l$ be such that $\overline{PQ} \cong \overline{QR}$. Now let $S \in$ be such that $\overline{PR} \cong \overline{RS}$. We get the picture,



We note that $90° + 2\alpha = 90° + \alpha + 2\beta = 180°$, and thus $2\beta = \alpha$. We can repeat this argument as many times as needed, until we get an angle that is less than θ.

Now we are ready to prove Playfair's axiom. So, let P be a point not on a line l.

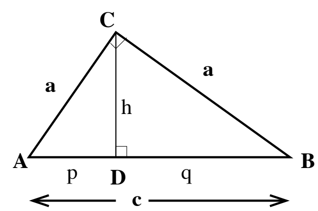
We first find a parallel m to l through P by constructing a common perpendicular n to l and m as we have done before (we call $Q = n \cap l$). Now take any other line m' through P. Since $m \neq m'$, then these lines form an angle α. We will work on the side of n where m' is 'below' m.



Using the mini lemma above we find a point $T \in l$ such that $\angle PTQ = \theta < \alpha$ (see picture above). Note that, since all triangles have angle-sum exactly 180° then $\angle QPT = 90° - \theta > 90° - \alpha$. It follows that the segment $\overline{PT}$ partitions the angle α. What we have found is a line that is 'above' m' and that intersects l. It follows that m' intersects l in some point between T and Q. Hence, there is a unique parallel to l through P. ■

Let us show that the Pythagorean theorem is equivalent to the fifth. We first notice that the Pythagorean theorem is proved in standard Euclidean geometry, and thus the only thing to show is that the Pythagorean theorem implies the fifth. We will do this via the angle sum of a triangle being 180^o.

Proof. Let $\triangle ABC$ be the isosceles right triangle below. We note that there are other smaller right triangles in the figure



Assuming the Pythagorean theorem we get,

$$2a^2 = c^2 \qquad p^2 + h^2 = a^2 \qquad h^2 + q^2 = a^2 \qquad c^2 = (p+q)^2 = p^2 + q^2 + 2pq$$

By plugging the last three equations into the first one we get

$$(h^2 + q^2) + (p^2 + h^2) = p^2 + q^2 + 2pq \qquad or \qquad \frac{h}{p} = \frac{q}{h}$$

If one sets $\frac{h}{p} = \frac{q}{h} = c$, then

$$h = cp \qquad hc = q$$

and we plug it into one of the Pythagorean expressions above to get

$$a^2 = (cp)^2 + (hc)^2 = c^2(p^2 + h^2) = c^2a^2$$

and thus $c = 1$. It follows that $h = p = q$. Hence, $\triangle ADC \cong \triangle CDB$ (by SAS) and both of them are isosceles. Since the four base angles are congruent and two of them add up to one right angle (looking at $\angle ACB$), then the base angles of $\triangle ABC$ also add to one right angle.

Hence, all isosceles right triangles have angle sum equal to two right angles. It follows from theorem 15 that all triangles have defect zero. ■