

## Aliasing

Digital spectrum analyzers work differently than analog spectrum analyzers. If you place an analog sinusoid at the input to an analog spectrum analyzer and if the frequency range displayed by the analyzer does not include the frequency of the analog input, you will not see any indication of the sinusoid in the display. This is not the case with a digital spectrum analyzer.

The spectrum mode of the PicoScope is a digital spectrum analyzer. An analog input signal is sampled (digitized) before the PicoScope begins processing this signal to determine its frequency content. If the input frequency is greater than the highest frequency in the display frequency range, the PicoScope (spectrum mode) will still show the signal, albeit at a frequency other than the true frequency. This phenomenon is known as aliasing. One can say that the input frequency accepts a new identity (alias) when it shows up on the display as a different frequency.

Consider an analog sinusoid, representing perhaps a carrier in a radio communications system,

$$\cos(2\pi f_c t)$$

The signal frequency is  $f_c$ . If this analog signal is sampled with a sampling frequency  $f_s$ , the time of the  $n$ -th sample is

$$t = nT_s$$

where the sampling period  $T_s$  (the time between adjacent samples) is the reciprocal of the sampling frequency,

$$T_s = \frac{1}{f_s}$$

and where  $n$  is an integer (index). The digital signal processor (DSP) will therefore see

$$\cos(2\pi f_c T_s n)$$

Figure 1 illustrates sampling of an analog sinusoid. In the case shown,  $f_s = 4f_c$ , so that there are four samples for each cycle of the analog sinusoid.

Now consider the case where a different analog sinusoid is sampled. In this case, the signal frequency is

$$f_c + kf_s$$

where  $k$  is an integer (any integer, positive or negative) and where  $f_s$  is the same sampling frequency as before. (For example, when  $k = -1$  the new signal frequency is less than the

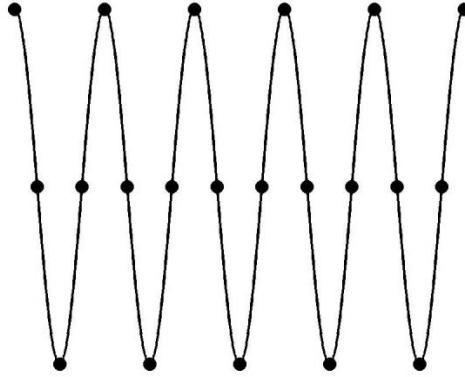


Figure 1: Sampling an analog sinusoid

original signal frequency by an amount equal to the sampling frequency.) The DSP will see

$$\cos[2\pi(f_c + kf_s)T_s n]$$

Since  $f_s T_s = 1$ , this expression can be written as

$$\begin{aligned} \cos[2\pi(f_c + kf_s)T_s n] &= \cos[2\pi f_c T_s n + 2\pi k f_s T_s n] \\ &= \cos[2\pi f_c T_s n + 2\pi k n] \\ &= \cos(2\pi f_c T_s n) \end{aligned}$$

The last step in the above mathematical development requires some explanation. If you add or subtract  $2\pi$  radians to the argument of a cosine, you don't change the value of the cosine. Above we subtracted  $2\pi$  radians  $kn$  times. (Note that  $kn$  is an integer since it is the product of two integers.)

The above equation tells us that if we sample a signal frequency  $f_c + kf_s$  (where  $k$  is any integer) with sampling frequency  $f_s$  the data passed to the DSP are identical to the data produced by sampling the signal frequency  $f_c$ . In other words, the DSP won't be able to tell the difference between an input analog frequency of  $f_c$  and an input analog frequency of  $f_c + kf_s$ , where  $k$  is any integer. Therefore, all of the signal frequencies  $f_c + kf_s$  (including  $f_c$ , corresponding to  $k = 0$ ) will show up at the same display frequency on a digital spectrum analyzer, such as the PicoScope in spectrum mode.

Figure 2 shows an example of aliasing. In this case,  $f_s = \frac{4}{5}f_c$ , so that there are 4 samples in every 5 cycles of the analog sinusoid (the solid curve in Figure 2). There are 5 cycles and 5 samples shown in Figure 2, but the last sample must be associated with the next set of 5 cycles, so there are only 4 samples associated with each set of 5 cycles. The relationship can also be written  $f_c = \frac{5}{4}f_s$ , which indicates that the analog sinusoid advances by one-and-one-quarter

cycles from one sample to the next. There is a second sinusoid shown in Figure 2; it corresponds to the frequency  $f_c - f_s$ . Since  $f_s = \frac{4}{5}f_c$  in this example, this second frequency equals  $\frac{1}{5}f_c$ ; this second frequency can also be written as  $\frac{1}{4}f_s$ . Notice how the data (the sampled points) lie on both sinusoids, so that a DSP given these data would have no way of knowing that it was an analog frequency of  $\frac{5}{4}f_s$  rather than an analog frequency of  $\frac{1}{4}f_s$  that provided these data.

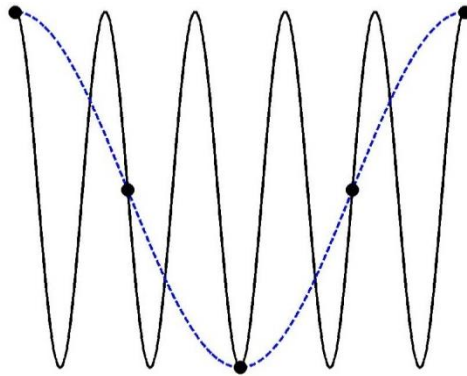


Figure 2: Illustration of aliasing

Another way of viewing aliasing is by means of this expression:

$$\cos(2\pi f_c t) = \text{Re}\{e^{j2\pi f_c t}\}$$

where  $\text{Re}\{\cdot\}$  denotes the real part of a complex value.

Figure 3 shows the complex plane (the Argand diagram). The horizontal axis represents the real part of a complex number, and the vertical axis the imaginary part. The complex exponential on the right-hand side of the above equation moves in a circle in a counter-clockwise direction with a uniform angular velocity. To sample the cosine is equivalent to sampling the complex exponential and then taking the real part of the sample. It is easy to see from the geometry of the circle that if we sample the complex exponential once for each one-and-one-quarter revolutions of the complex point, then we get exactly the same sample values as if we were sampling only once every one-quarter revolution of the complex point.

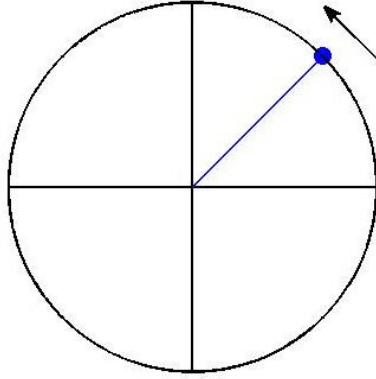


Figure 3: Complex exponential

The display range of a digital spectrum analyzer is always chosen to be 0 to  $f_s/2$ . If the analog signal frequency  $f_c$  at the input of the spectrum analyzer already lies within the display range (0 to  $f_s/2$ ), then we say there is no aliasing. On the other hand, if  $f_c$  is larger than  $f_s/2$ , there will be an alias frequency  $f_{\text{alias}}$  in the range

$$0 \leq f_{\text{alias}} \leq f_s/2$$

and that is the frequency that will display on the digital spectrum analyzer. Two different but equivalent procedures for calculating  $f_{\text{alias}}$  are described below.

### Method 1 for Finding $f_{\text{alias}}$

We assume  $f_c > f_s/2$ . Otherwise, there is no need to calculate an alias frequency. Calculate the difference  $f_x = f_c - f_s$ . If  $f_x$  is larger than  $f_s/2$ , then keep subtracting  $f_s$  from  $f_c$  until a frequency  $f_x$  is reached that is less than  $f_s/2$ . If  $f_x$  is between 0 and  $f_s/2$ , then  $f_{\text{alias}} = f_x$ . If  $f_x$  is between  $-f_s/2$  and 0, then  $f_{\text{alias}} = -f_x$ . In this way,  $f_{\text{alias}}$  is always positive and less than  $f_s/2$ .

Consider the example of a 100 kHz carrier frequency that will be viewed with the PicoScope (spectrum mode). Suppose the display range of 195 kHz is selected on the PicoScope. This is  $f_s/2$ , so the sampling frequency is 390 kHz. The 100 kHz lies already within the display range, so there is no aliasing in this case.

Now suppose we change the display range to 98 kHz. The sampling frequency now automatically changes to 196 kHz. The 100 kHz of our carrier is outside the display range, so we know that we will see an alias. If we subtract 196 kHz ( $f_s$ ) from 100 kHz, we will get  $-96$  kHz. Corresponding to this negative frequency is a positive frequency 96 kHz. This is the alias frequency. Our carrier will show up on the PicoScope with the correct signal level but at the alias frequency 96 kHz.

Finally, we consider what happens if we change the display range to 49 kHz. Of course, there will be an alias in this case as well. Will it be the same alias frequency? No! The sampling frequency changes automatically to 98 kHz when we select a display range of 49 kHz. If we subtract 98 kHz ( $f_s$ ) from 100 kHz, we will get 2 kHz. This is the new alias frequency.

Let us summarize the above example. We send a 100-kHz analog sinusoid to the PicoScope. When the display range is 195 kHz, the display shows it at 100 kHz. If we switch the display range to 98 kHz, the signal suddenly “moves” to 96 kHz. If we then switch the display range to 49 kHz, the signal “moves” again, this time to 2 kHz. The displayed signal level, in decibels, is the same (and correct) for all settings of the display range.

The rationale for this method is as follows. The frequencies  $f_c + kf_s$  (for  $k$  any integer, positive or negative) are evenly spaced along the frequency axis  $-\infty < f < \infty$ , each frequency distant by  $f_s$  from its nearest neighbor on either side. Every frequency in this set will result in the same set of samples when using the sampling frequency  $f_s$ . From this set of frequencies ( $f_c + kf_s$ ) there will always be one and only one frequency  $f_x = f_c + k_x f_s$  (where  $k_x$  is a negative integer) that lies within the range

$$-f_s/2 \leq f_x \leq f_s/2$$

$f_x$  can be either positive or negative, but  $f_{\text{alias}}$  is defined to be positive. So  $f_{\text{alias}} = |f_x|$ .

The justification for saying  $f_{\text{alias}} = |f_x|$  comes from Euler’s identities,

$$\begin{aligned}\cos(2\pi ft) &= \frac{1}{2} e^{j2\pi ft} + \frac{1}{2} e^{-j2\pi ft} \\ \sin(2\pi ft) &= \frac{1}{2j} e^{j2\pi ft} - \frac{1}{2j} e^{-j2\pi ft}\end{aligned}$$

That is to say, every real sinusoid always consists of a negative-and-positive frequency pair.

### Method 2 for Finding $f_{\text{alias}}$

In this second method for calculating  $f_{\text{alias}}$  we will think in terms of the displayed frequency range

$$f_R = f_s/2$$

for a digital spectrum analyzer. We assume  $f_c > f_R$ . Otherwise, there is no need to calculate an alias frequency.

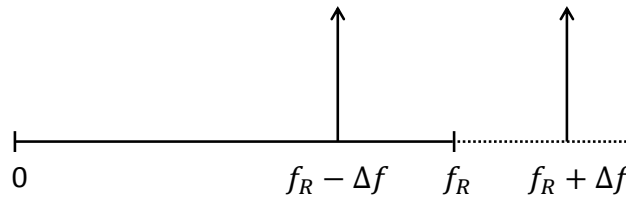
Consider first the case where  $f_R < f_c < 2f_R$ . Imagine a point at DC ( $f = 0$ ) on the frequency axis. If we were using an *analog* spectrum analyzer, we would envision the point moving  $f_c$  Hz to the

right before stopping to mark the location of the analog frequency. However, with a *digital* spectrum analyzer, employing a sampled signal, we can move the point from the origin only  $f_R$  Hz to the right. In order to complete the required travel of  $f_c$  Hz, we imagine the point “bouncing off the wall” at  $f_R$  Hz and then moving back to the left by an additional

$$\Delta f = f_c - f_R \text{ Hz}$$

The total travel is therefore  $f_R + \Delta f = f_c$  Hz, as required, and the point comes to rest at  $f_R - \Delta f$ . In other words,

$$f_{\text{alias}} = f_R - \Delta f, \quad f_R < f_c < 2f_R$$



This can also be written as  $f_{\text{alias}} = f_s - f_c$ , which is the same result as given by Method 1 for the case  $f_s/2 < f_c < f_s$ .

This method can be applied also when  $f_c > 2f_R$ , but more than one bounce is then required. For example, if  $f_c$  is a little larger than  $2f_R$ , we envision moving the point from DC to the right, where it bounces off the wall at  $f_R$  Hz, then to the left, where it bounces off the wall at 0 Hz, and then to the right again, reaching its final resting place a little to the right of 0 Hz. So the alias is  $f_{\text{alias}}$  when  $f_c = 2f_R + f_{\text{alias}}$  (where  $0 < f_{\text{alias}} < f_R$ ). With this method, you can use as many bounces as necessary to achieve the total travel of  $f_c$  Hz.

Another way of thinking about this aliasing is to say that the frequency axis folds at  $f_R$ , so that the analog frequency  $f_R + \Delta f$  appears at  $f_R - \Delta f$ .

We now use this second method for the example considered earlier.  $f_c = 100$  kHz and, initially,  $f_R = 98$  kHz. We envision a point on the frequency axis that leaves DC and moves 98 kHz to the right, where it bounces off the wall at 98 kHz and then moves back to the left by 2 kHz, for a total travel of 100 kHz. The final resting place is  $f_{\text{alias}} = 96$  kHz.

Now we consider  $f_c = 100$  kHz and  $f_R = 49$  kHz. In this case, we envision a point on the frequency axis that leaves DC and moves to the right, bouncing off the wall at 49 kHz, then

moves to the left and bounces off the wall at 0 Hz. The total travel is 100 kHz, and the final resting place is  $f_{\text{alias}} = 2 \text{ kHz}$ .

Some people consider this second method more intuitive and simpler than the first. However, if the number of bounces gets large, the second method becomes awkward. So the first method is generally preferred when  $f_c$  is much larger than  $f_R$ .

The rationale for this method is as follows. Suppose that the analog frequency equals  $f_R + \Delta f$ , where  $0 < \Delta f < f_R$ . The real sinusoid with this frequency has a Fourier transform with two Dirac delta functions: one at  $f_R + \Delta f$  and one at  $-(f_R + \Delta f)$ . The second of these Dirac delta functions aliases to  $2f_R - (f_R + \Delta f) = f_R - \Delta f$ .