1. Let \( A \subseteq \mathbb{R} \) with \( m^*(A) > 0 \). Let \( C \) be a choice set for \( A \). Prove that \( C \) is not measurable.

**Proof**: First, we prove that

\[
A \subseteq \bigcup_{q \in \mathbb{Q}} (C + q)
\]

Indeed, let \( x \in A \). Since \( C \) is a choice set for \( A \), we have that \( x \sim c \) for some (unique) \( c \in C \). Hence \( x - c \in \mathbb{Q} \) by definition of the rational equivalent relation. So \( x = c + (x - c) \in C + (x - c) \).

Using monotonicity, countable subadditivity and the fact that the outer measure is translation invariant, we get that

\[
0 < m^*(A) \leq m^* \left( \bigcup_{q \in \mathbb{Q}} (C + q) \right) \leq \sum_{q \in \mathbb{Q}} m^*(C + q) = \sum_{q \in \mathbb{Q}} m^*(C) = \begin{cases} 
0 & \text{if } m^*(C) = 0 \\
+\infty & \text{if } m^*(C) > 0
\end{cases}
\]

Since \( m^*(A) > 0 \), we must have that \( m^*(C) > 0 \).

Suppose that \( m^*(C \cap [n, n+1)) = 0 \) for all \( n \in \mathbb{Z} \). Since \( C = \cup_{n \in \mathbb{Z}} (C \cap [n, n+1)) \), it follows from countable subadditivity that

\[
m^*(C) = m^* \left( \bigcup_{n \in \mathbb{Z}} (C \cap [n, n+1)) \right) \leq \sum_{n \in \mathbb{Z}} m^* (C \cap [n, n+1)) = 0
\]

a contradiction.

Hence \( m^*(C \cap [n, n+1)) > 0 \) for some \( n \in \mathbb{Z} \). Note that \( C \cap [n, n+1) \) is a bounded set whose outer measure is strictly positive. By Theorem 2.25, any choice set for \( C \cap [n, n+1) \) is non-measurable. However, since \( C \) was a choice set (for \( A \)), we get that no two elements in \( C \cap [n, n+1) \) are related under the rational equivalent relation.

This implies that \( C \) is non-measurable: if \( C \) was measurable, then \( C \cap [n, n+1) \) would be measurable (since \( [n, n+1) \) is measurable), a contradiction.

2. Let \( D \subseteq \mathbb{R} \) be measurable and \( f : D \to \mathbb{R} \) a real-valued function. Prove that \( f \) is measurable if and only if \( f^{-1}(B) \) is measurable for any Borel set \( B \).

**Proof**: Suppose first that \( f^{-1}(B) \) is measurable for each Borel set \( B \). Since open sets are Borel sets, we get that \( f^{-1}(O) \) is measurable for each open set \( O \). So \( f \) is measurable by Theorem 3.4.

Suppose next that \( f \) is measurable. Put \( \mathcal{B} = \{ B \subseteq \mathbb{R} \mid f^{-1}(B) \text{ is measurable} \} \).

We claim that \( \mathcal{B} \) is a \( \sigma \)-algebra (note that this is true whenever \( D \) is measurable; \( f \) does not need to be measurable). Pick \( B \in \mathcal{B} \). Then

\[
f^{-1}(\widetilde{B}) = f^{-1}(B) = D \setminus f^{-1}(B)
\]

which is a difference of measurable sets and hence is measurable. So \( \widetilde{B} \in \mathcal{B} \).

Pick \( B_n \in \mathcal{B} \) for all \( n \geq 1 \). Then

\[
f^{-1}(\bigcup_{n=1}^{+\infty} B_n) = \bigcup_{n=1}^{+\infty} f^{-1}(B_n)
\]

which is a countable union of measurable sets and hence is measurable. So \( \bigcup_{n=1}^{+\infty} B_n \in \mathcal{B} \).

Hence \( \mathcal{B} \) is a \( \sigma \)-algebra, which proves the claim.
Since \( f \) is measurable, it follows from Theorem 3.4 that \( f^{-1}(O) \) is measurable for each open set \( O \). Hence \( O \in \mathcal{B} \) for each open set \( O \). Since the collection of all Borel sets is the smallest \( \sigma \)-algebra containing the collection of all the open sets and \( \mathcal{B} \) is some \( \sigma \)-algebra containing the collections of all the open sets, we get that \( B \in \mathcal{B} \) for each Borel set \( B \). Hence \( f^{-1}(B) \) is measurable for each Borel set \( B \).

3. True/False : If \( f : [a, b] \to \mathbb{R} \) is continuous a.e. on \([a, b]\), then there exists a continuous function \( g : [a, b] \to \mathbb{R} \) such that \( f = g \) a.e. on \([a, b]\).

**Solution** : **FALSE**

Put

\[
    f : [-1, 1] \to \mathbb{R} : x \to \begin{cases} 
    -1 & \text{if } -1 \leq x \leq 0 \\
    1 & \text{if } 0 < x \leq 1 
\end{cases}
\]

Then \( f \) is continuous over \([-1, 0) \cup (0, 1]\). So \( f \) is continuous a.e. on \([-1, 1]\). Suppose that \( g : [-1, 1] \to \mathbb{R} \) is continuous over \([-1, 1]\) and \( f = g \) a.e. on \([-1, 1]\). Put \( E = \{x \in [-1, 1] \mid f(x) \neq g(x)\} \). Then \( m(E) = 0 \).

Pick \( n \in \mathbb{N} \). If \((0, \frac{1}{n}) \setminus E = \emptyset\), then \((0, \frac{1}{n}) \subseteq E\), a contradiction since \( m\left((0, \frac{1}{n})\right) = \frac{1}{n} > 0 = m(E)\). So we can pick \( a_n \in (0, \frac{1}{n}) \setminus E \). Since \( 0 < a_n < \frac{1}{n} \) for all \( n \in \mathbb{N} \), we get that \( \{a_n\}_{n \geq 1} \to 0 \). Since \( g \) is continuous at \( 0 \), we get that \( \{g(a_n)\}_{n \geq 1} \to g(0) \). Since \( a_n \notin E \), \( g(a_n) = f(a_n) = 1 \) for all \( n \geq 1 \). So \( \{g(a_n)\}_{n \geq 1} = \{1\}_{n \geq 1} \to 1 \). Hence \( g(0) = 1 \).

Similarly, we can pick \( b_n \in (-\frac{1}{n}, 0) \setminus E \) for all \( n \in \mathbb{N} \). Then \( \{b_n\}_{n \geq 1} \to 0 \) and so \( \{g(b_n)\}_{n \geq 1} \to g(0) \). But \( g(b_n) = f(b_n) = -1 \) and so \( \{g(b_n)\}_{n \geq 1} \to -1 \). Hence \( g(0) = -1 \), a contradiction.

4. Let \( f : [a, b] \to \mathbb{R} \) and \( g : [a, b] \to \mathbb{R} \) be continuous on \([a, b]\) such that \( f = g \) a.e. on \([a, b]\). Prove that \( f = g \) on \([a, b]\).

**Proof** : Suppose that \( f \neq g \) on \([a, b]\). Let \( x_0 \in [a, b] \) with \( f(x_0) \neq g(x_0) \). Put \( h = f - g \). Then \( h \) is continuous on \([a, b]\) and \( h(x_0) \neq 0 \). Since \( h \) is continuous at \( x_0 \), we know that \( h \neq 0 \) in some interval containing \( x_0 \):

\[
    \exists \delta > 0 : \forall x \in [a, b] : |x - x_0| < \delta \implies h(x) \neq 0
\]

So \( h \neq 0 \) on \([a, b] \cap (x_0 - \delta, x_0 + \delta)\). Hence \( g \neq f \) on \([a, b] \cap (x_0 - \delta, x_0 + \delta)\), a contradiction since \( f = g \) a.e. on \([a, b]\) and \( m([a, b] \cap (x_0 - \delta, x_0 + \delta)) > 0 \).

So \( f = g \) on \([a, b]\).

5. Let \( I \) be an interval and \( f : I \to \mathbb{R} \) a function that is monotone on \( I \). Then \( f \) is measurable.

**Proof** : First we prove the following characterization of an interval (basically ‘no wholes’):

Let \( S \subseteq \mathbb{R} \) such that for all \( a, b \in S \) with \( a < b \), we have that \( (a, b) \subseteq S \). Then \( S \) is an interval.

If \( |S| = 0 \) or \( |S| = 1 \) then \( S \) is an interval. So we may assume that \( |S| \geq 2 \).

Put \( \alpha = \inf S \) and \( \beta = \sup S \). Note that it is possible that \( \alpha = -\infty \) and/or \( \beta = +\infty \). Clearly, \( S \subseteq [\alpha, \beta] \) since \( \alpha \leq s \leq \beta \) for all \( s \in S \). We show that \( (\alpha, \beta) \subseteq S \). Pick \( x \in (\alpha, \beta) \). So \( \inf S < x < \sup S \). Hence \( a < x < b \) for some \( a, b \in S \). Thus \( x \in (a, b) \subseteq S \). So \( (\alpha, \beta) \subseteq S \). So we have

\[
    (\alpha, \beta) \subseteq S \subseteq [\alpha, \beta]
\]

That gives us four possibilities for \( S \):

\[
    S = (\alpha, \beta) \text{ or } S = (\alpha, \beta) \text{ or } S = [\alpha, \beta] \text{ or } S = [\alpha, \beta]
\]
Hence $S$ is an interval, which proves the characterization of an interval.

Now we can prove that $f$ is measurable. WLOG, assume that $f$ is non-decreasing on $I$. Let $\alpha \in \mathbb{R}$. Put $S = \{ x \in I : f(x) > \alpha \}$. If $|S| \leq 1$ then $S$ is measurable. So assume $|S| \geq 2$. Let $a, b \in S$ with $a < b$. Then $f(a) > \alpha$. Let $x \in (a, b)$. Then $x \in I$ and $a < x$. Hence $\alpha < f(a) \leq f(x)$. So $f(x) > \alpha$ and $x \in S$. Thus $(a, b) \subseteq S$. By the characterization of an interval, we get that $S$ is an interval. So $S$ is measurable. Hence $f$ is measurable. \qed