

1. Let C be the binary code word generator matrix $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$.

- (a) Set up a standard array for C .
 (b) Write down the coset leaders of each coset.
 (c) Use your standard array to decode the following words :
- 1111
 - 1110
 - 1001

Solution : (a) First, we find all the codewords. Recall that the codewords are all the linear combinations of the rows of the generator matrix. Hence

$$C = \{0000, 1010, 0101, 1111\}$$

This is the first row of the standard array.

Next, we fill the standard array : we find a word \mathbf{x} that is not in the array yet and add the coset $\mathbf{x} + C$ as a new row to the standard array. We make sure that a coset leader is in the first column. Let's say we start with $\mathbf{x} = 1000$. Then we get

$$\begin{array}{cccc} 0000 & 1010 & 0101 & 1111 \\ 1000 & 0010 & 1101 & 0111 \end{array}$$

We continue with $\mathbf{x} = 0100$ and find

$$\begin{array}{cccc} 0000 & 1010 & 0101 & 1111 \\ 1000 & 0010 & 1101 & 0111 \\ 0100 & 1110 & 0001 & 1011 \end{array}$$

Finally, we pick any remaining word, say $\mathbf{x} = 1100$ and get the following standard array :

$$\begin{array}{cccc} 0000 & 1010 & 0101 & 1111 \\ 1000 & 0010 & 1101 & 0111 \\ 0100 & 1110 & 0001 & 1011 \\ 1100 & 0110 & 1001 & 0011 \end{array}$$

It's important to realize that this array is not unique. We could have picked 0010 as choice for a coset leader. Then the second row would have been

$$0010 \quad 1000 \quad 0111 \quad 1101$$

(b) Using our standard array, we easily get

Coset	Coset Leaders
$\{0000, 1010, 0101, 1111\}$	0000
$\{1000, 0010, 1101, 0111\}$	1000, 0010
$\{0100, 1110, 0001, 1011\}$	0100, 0001
$\{1100, 0110, 1001, 0011\}$	1100, 0110, 1001, 0011

Note that this is independent of the standard array we choose.

(c) Decoding using a standard array is quite simple : we look up the received word in the standard array and decode as the codeword (in the first row) that is in the same column as the received word. This clearly depends on the standard array we choose. We get

We decode 1111 as 1111.

We decode 1110 as 1010.

We decode 1001 as 0101.

2. Let C be the binary code with parity check matrix $\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$.

(a) Set up a syndrome table for C .

(b) Use your syndrome table to decode the word 11010.

Solution : We pick words of small weight (first weight zero, then weight one, then weight two,...) and evaluate their syndrome until we get every single syndrome exactly once. We find

coset leader	syndrome
00000	000
10000	100
01000	010
00100	001
00010	111
00001	101
01100	011
11000	110

Note that the first six rows are unique since these cosets have exactly one coset leader. The last two rows are not unique since those cosets have more than one coset leader. We could have chosen 10010 instead of 01100 in the seventh row.

(b) First, we calculate

$$\text{syn}(11010) = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 001$$

Next, we look up this syndrome in the syndrome table. We decode by subtracting the coset leader from the received word. So we decode 11010 as

$$11010 - 00100 = 11110$$

For this particular received word, the decoding does not depend on the syndrome table as we used the fourth row (a coset with a unique coset leader). \square

3. Let C be a binary $[n, k]$ -code. Fix $1 \leq i \leq n$. Let C_i be the set of all codewords in C whose i -th digit is zero. So

$$C_i = \{x_1 \dots x_n \in C : x_i = 0\}$$

- (a) Prove that C_i is linear.
 (b) Prove that C_i is a subgroup of C of index at most two.
 (c) Deduce that exactly one of the following holds:
- The i -th digit in every codeword in C is zero.
 - The i -th digit in exactly half of the codewords is zero.

Proof : (a) Let $\mathbf{x}, \mathbf{y} \in C_i$ and $a, b \in \{0, 1\}$. Looking at the i -position, we see that

$$(a\mathbf{x} + b\mathbf{y})_i = ax_i + by_i = a \cdot 0 + b \cdot 0 = 0$$

Hence $a\mathbf{x} + b\mathbf{y} \in C_i$. So C_i is linear.

(b) Recall that the index of C_i in C (notation : $[C : C_i]$) is the number of cosets of C_i in C .

If $C_i = C$ then C_i is the only coset of C_i in C and $[C : C_i] = 1$.

So we may assume that $C_i \neq C$. Fix $\mathbf{y} \in C \setminus C_i$. So \mathbf{y} has a one in the i -th position. We claim that $\mathbf{y} + C_i$ is the set of all codewords whose i -th digit is one. Pick $\mathbf{x} \in \mathbf{y} + C_i$. Then $\mathbf{x} = \mathbf{y} + \mathbf{c}$ for some $\mathbf{c} \in C_i$. Hence $\mathbf{x} \in C$ since $\mathbf{y}, \mathbf{c} \in C$ and $x_i = (\mathbf{y} + \mathbf{c})_i = y_i + c_i = 1 + 0 = 1$. Pick $\mathbf{u} \in C$ with a one in the i -th position. Put $\mathbf{v} = \mathbf{u} - \mathbf{y}$. Then $\mathbf{v} \in C$ since $\mathbf{u}, \mathbf{y} \in C$ and $v_i = (\mathbf{u} - \mathbf{y})_i = u_i - y_i = 1 - 1 = 0$. So $\mathbf{v} \in C_i$ and $\mathbf{u} = \mathbf{y} + (\mathbf{u} - \mathbf{y}) = \mathbf{y} + \mathbf{v} \in \mathbf{y} + C_i$, which proves our claim.

Since the code C is binary, we have that every codeword has either a zero or a one in the i -th position. Hence $C = C_i \cup (\mathbf{y} + C_i)$. So C is the disjoint union of two cosets of C_i . Since the cosets of C_i in C form a partition of C , we must have that C_i has exactly two cosets in C . So $[C : C_i] = 2$

(c) From (b), it follows that $[C : C_i] \in \{1, 2\}$.

If $[C : C_i] = 1$ then $C_i = C$ and so every codeword has a zero in the i -th position.

So we may assume that $[C : C_i] = 2$. Since every coset of C_i in C has the same number of elements (namely $|C_i|$) and C_i has two cosets in C , we get that $|C_i| = \frac{1}{2}|C|$. So exactly half of the codewords have a zero in the i -th position. Since the code is binary, we get that exactly half of the codewords have a one in the i -th position. \square

4. Let C be the binary code of length 16 containing the words $x_1x_2 \dots x_{15}x_{16}$ such that in the matrix

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ x_5 & x_6 & x_7 & x_8 \\ x_9 & x_{10} & x_{11} & x_{12} \\ x_{13} & x_{14} & x_{15} & x_{16} \end{bmatrix}$$

every row and every column contains an even number of ones.

- (a) Prove that C is a linear code.
 (b) Find the dimension and minimum distance of C .
 (c) List all possibilities to decode the following words, using Nearest Neighbor Decoding:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution : (a) Let $\mathbf{x}, \mathbf{y} \in C$ and $\alpha, \beta \in \{0, 1\}$. We need to prove that $\alpha\mathbf{x} + \beta\mathbf{y} \in C$. Since $\mathbf{0} \in C$, we may assume that $\alpha = \beta = 1$. Let \mathbf{x}_1 be the first row of \mathbf{x} and \mathbf{y}_1 the first row of \mathbf{y} . Then the first row of $\mathbf{x} + \mathbf{y}$ is $\mathbf{x}_1 + \mathbf{y}_1$. By HW 2 Ex. #2, we have that

$$w(\mathbf{x}_1 + \mathbf{y}_1) = w(\mathbf{x}_1) + w(\mathbf{y}_1) - 2w(\mathbf{x}_1 * \mathbf{y}_1)$$

Since $\mathbf{x}, \mathbf{y} \in C$, we get that $w(\mathbf{x}_1)$ and $w(\mathbf{y}_1)$ are even. Hence $w(\mathbf{x}_1 + \mathbf{y}_1)$ is even. Similarly, we get that the weight of any row and any column of $\mathbf{x} + \mathbf{y}$ is even. So $\mathbf{x} + \mathbf{y} \in C$.

Hence C is linear.

(b) Consider the following map from C to the set of all 3×3 -matrices :

$$\theta : C \rightarrow \{0, 1\}^{3 \times 3} : \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ x_5 & x_6 & x_7 & x_8 \\ x_9 & x_{10} & x_{11} & x_{12} \\ x_{13} & x_{14} & x_{15} & x_{16} \end{bmatrix} \rightarrow \begin{bmatrix} x_1 & x_2 & x_3 \\ x_5 & x_6 & x_7 \\ x_9 & x_{10} & x_{11} \end{bmatrix}$$

We prove that θ is a bijection.

Let $\mathbf{x}, \mathbf{y} \in C$ with $\theta(\mathbf{x}) = \theta(\mathbf{y})$. Recall that the weight of every row and every column of a codeword is even. So

$$x_4 = x_1 + x_2 + x_3 = y_1 + y_2 + y_3 = y_4$$

Similarly, we get that

$$x_8 = y_8, \quad x_{12} = y_{12}, \quad x_{13} = y_{13}, \quad x_{14} = y_{14} \quad \text{and} \quad x_{15} = y_{15}$$

For x_{16} and y_{16} , we get

$$x_{16} = x_{13} + x_{14} + x_{15} = y_{13} + y_{14} + y_{15} = y_{16}$$

So $\mathbf{x} = \mathbf{y}$ and θ is one-to-one.

Let $\begin{bmatrix} x_1 & x_2 & x_3 \\ x_5 & x_6 & x_7 \\ x_9 & x_{10} & x_{11} \end{bmatrix} \in \{0, 1\}^{3 \times 3}$. Put

$$\begin{aligned} x_4 &= x_1 + x_2 + x_3 \\ x_8 &= x_5 + x_6 + x_7 \\ x_{12} &= x_9 + x_{10} + x_{11} \\ x_{13} &= x_1 + x_5 + x_9 \\ x_{14} &= x_2 + x_6 + x_{10} \\ x_{15} &= x_3 + x_7 + x_{11} \\ x_{16} &= x_{13} + x_{14} + x_{15} \end{aligned}$$

Then $\mathbf{x} = x_1x_2 \dots x_{15}x_{16}$ is a binary word of length 16 such that the weight of every row and the first three columns is even. For the fourth column, we find

$$\begin{aligned} x_{16} &= x_{13} + x_{14} + x_{15} \\ &= (x_1 + x_5 + x_9) + (x_2 + x_6 + x_{10}) + (x_3 + x_7 + x_{11}) \\ &= (x_1 + x_2 + x_3) + (x_5 + x_6 + x_7) + (x_9 + x_{10} + x_{11}) \\ &= x_4 + x_8 + x_{12} \end{aligned}$$

So the fourth column has even weight as well. Hence $\mathbf{x} \in C$ and $\theta(\mathbf{x}) = \begin{bmatrix} x_1 & x_2 & x_3 \\ x_5 & x_6 & x_7 \\ x_9 & x_{10} & x_{11} \end{bmatrix}$. So θ is onto.

There is another way to see that the weight of the fourth column is even. As the weight of each row of \mathbf{x} is even, we get that the weight of \mathbf{x} is even (it is the sum of the weights of the four rows). So the sum of the weights

of the four columns of \mathbf{x} is also even. As the weights of the first three columns of \mathbf{x} are even, the weight of the fourth column must be even as well.

So θ is a bijection. Since $|\{0, 1\}^{3 \times 3}| = 2^9$, we get that $|C| = 2^9$. Hence $\dim(C) = 9$.

It's relatively easy to see that $w(C) = 4$. Clearly, $w(\mathbf{x})$ is even for all $\mathbf{x} \in C$. Let $\mathbf{0} \neq \mathbf{x} \in C$. Then \mathbf{x} has at least one non-zero entry, say in the i -th row and the j -th column. As the weight of the i -th row is even, there must be another non-zero entry in the i -th row (besides the (i, j) -entry). As the weight of the j -th column is even, there must be another non-zero entry in the j -th column (besides the (i, j) -entry). Hence $w(\mathbf{x}) \geq 3$. Since $w(\mathbf{x})$ is even, we get that $w(\mathbf{x}) \geq 4$.

Thus $w(C) \geq 4$. Since $1100110000000000 \in C$, we have $d(C) = w(C) = 4$.

(c) We start with a general remark. Recall from HW 2 #6 that for binary vectors \mathbf{x} and \mathbf{y} , we have

$$w(\mathbf{x} + \mathbf{y}) = w(\mathbf{x}) + w(\mathbf{y}) - 2w(\mathbf{x} * \mathbf{y})$$

Since $d(\mathbf{x}, \mathbf{y}) = w(\mathbf{x} - \mathbf{y}) = w(\mathbf{x} + \mathbf{y})$, we get

$$d(\mathbf{x}, \mathbf{y}) \equiv w(\mathbf{x}) + w(\mathbf{y}) \pmod{2} \quad (*)$$

We will use this where \mathbf{x} and \mathbf{y} are binary vectors of length 16 or rows or column of a 4×4 -matrix.

For the first received word, put $\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. Since $w(\mathbf{y}) = 3$, $\mathbf{y} \notin C$. Since $\mathbf{c} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \in C$

and $d(\mathbf{y}, \mathbf{c}) = 1$, we will decode \mathbf{y} as any codeword at distance one from \mathbf{y} . Let $\mathbf{x} \in C$ with $d(\mathbf{y}, \mathbf{x}) = 1$. Note that $\mathbf{x} \neq \mathbf{0}$ since $d(\mathbf{y}, \mathbf{0}) = 3$. The second row of \mathbf{x} has even weight. If the second row of \mathbf{x} contains two or more ones, then $d(\mathbf{x}, \mathbf{y}) \geq 2$, a contradiction. So the second row of \mathbf{x} contains all zeros. Similarly, the third row, the second column and the third column of \mathbf{x} contain all zeros. Since $\mathbf{x} \neq \mathbf{0}$, we must have that

$$\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \mathbf{c}$$

So we decode $\mathbf{y} = 1001000000001000$ as $\mathbf{c} = 1001000000001001$.

For the second received word, put $\mathbf{y} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Then $w(\mathbf{y}) = 2$ and so $\mathbf{y} \notin C$. If $\mathbf{c} \in C$ then $d(\mathbf{y}, \mathbf{c})$ is

even by (*). Clearly, $\mathbf{0} \in C$ and $d(\mathbf{y}, \mathbf{0}) = 2$. Hence we will decode \mathbf{y} as any codeword at distance two from \mathbf{y} . Let $\mathbf{0} \neq \mathbf{x} \in C$ with $d(\mathbf{y}, \mathbf{x}) = 2$. By the Triangle inequality, we get

$$w(\mathbf{x}) \leq w(\mathbf{x} - \mathbf{y}) + w(\mathbf{y}) = 2 + 2 = 4$$

So $w(\mathbf{x}) = 4$ since $\mathbf{x} \neq \mathbf{0}$ and $w(C) = 4$. For $1 \leq j \leq 4$, let \mathbf{x}_j be the j -th column of \mathbf{x} and \mathbf{y}_j the j -th column of \mathbf{y} . As $w(\mathbf{x}_1)$ is even ($\mathbf{x} \in C$) and $w(\mathbf{y}_1) = 1$, it follows from (*) that $d(\mathbf{x}_1, \mathbf{y}_1)$ is odd. Since $d(\mathbf{x}, \mathbf{y}) = 2$, we must have that $d(\mathbf{x}_1, \mathbf{y}_1) = 1$. Similarly, $d(\mathbf{x}_2, \mathbf{y}_2) = 1$. Since $d(\mathbf{x}, \mathbf{y}) = 2$, we must have that $d(\mathbf{x}_3, \mathbf{y}_3) = d(\mathbf{x}_4, \mathbf{y}_4) = 0$. If $w(\mathbf{x}_1) = 4$ or $w(\mathbf{x}_2) = 4$ then $w(\mathbf{x}) \geq 8$ since every row of \mathbf{x} must have even weight, a contradiction. Since $w(\mathbf{x}) = 4$, we get that $w(\mathbf{x}_1) = w(\mathbf{x}_2) = 2$. Since $d(\mathbf{x}_1, \mathbf{y}_1) = d(\mathbf{x}_2, \mathbf{y}_2) = 1$, this implies that the first row of \mathbf{x} is 1100. Hence there are three possibilities for \mathbf{x} and we decode

$$\mathbf{y} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ as } \mathbf{c} \in \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \right\}$$

For the third received word, put $\mathbf{y} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Then $w(\mathbf{y}) = 3$ and so $\mathbf{y} \notin C$. For $\mathbf{z} \in \{0,1\}^{16}$ and

$1 \leq j \leq 4$, we denote by \mathbf{z}_j the j -th column of \mathbf{z} (as a 4×4 -matrix). Let $\mathbf{c} \in C$. By (*), $d(\mathbf{y}, \mathbf{c})$ is odd. Again by (*), $d(\mathbf{y}_j, \mathbf{c}_j)$ is odd (and hence at least 1) for $j = 1, 2, 3$. So $d(\mathbf{y}, \mathbf{c}) \geq 3$. Since $\mathbf{0} \in C$ and $d(\mathbf{y}, \mathbf{0}) = 3$, we will decode \mathbf{y} as any codeword at distance three from \mathbf{y} . Let $\mathbf{0} \neq \mathbf{x} \in C$ with $d(\mathbf{y}, \mathbf{x}) = 3$. By the above, $d(\mathbf{y}_j, \mathbf{x}_j) = 1$ for $j = 1, 2, 3$ and so $d(\mathbf{y}_4, \mathbf{x}_4) = 0$. Thus $\mathbf{x}_4 = \mathbf{y}_4$. It follows from the Triangle Inequality that

$$w(\mathbf{x}) \leq w(\mathbf{x} - \mathbf{y}) + w(\mathbf{y}) = 3 + 3 = 6$$

Since $w(C) = 4$ and $w(\mathbf{x})$ is even, we get that $w(\mathbf{x}) = 4$ or $w(\mathbf{x}) = 6$. If $w(\mathbf{x}_j) = 4$ for some $1 \leq j \leq 4$ then $w(\mathbf{x}) \geq 8$, a contradiction. So $w(\mathbf{x}_j) \in \{0, 2\}$ for $1 \leq j \leq 4$.

Suppose first that $w(\mathbf{x}) = 4$. Then $w(\mathbf{x}_j) = 0$ for exactly one element from $\{1, 2, 3\}$ and the ones in the other columns of \mathbf{y} show up in \mathbf{x} (as $d(\mathbf{y}_j, \mathbf{x}_j) = 1$ for $j = 1, 2, 3$). This gives us five possibilities for \mathbf{x} :

$$\mathbf{x} \in \left\{ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \right\}$$

Suppose next that $w(\mathbf{x}) = 6$. Then $w(\mathbf{x}_j) = 2$ for $j = 1, 2, 3$ and all the ones in \mathbf{y} show up in \mathbf{x} (as $d(\mathbf{y}_j, \mathbf{x}_j) = 1$ for $j = 1, 2, 3$). So \mathbf{x} is of the form

$$\begin{bmatrix} 1 & 1 & \cdot & 0 \\ \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & 1 & 0 \\ \cdot & \cdot & \cdot & 0 \end{bmatrix}$$

Since the rows of \mathbf{x} have even weight, we see that the first and third row of \mathbf{x} have weight two. As $w(\mathbf{x}) = 6$, either row two has weight two and row four has weight zero or vice versa. Once we have chosen which row has weight zero, we can complete the third column of \mathbf{x} . We get four possibilities for \mathbf{x} :

$$\mathbf{x} \in \left\{ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right\}$$

So there are ten possibilities (codewords at distance three from \mathbf{y}) to decode \mathbf{y} : $\mathbf{0}$, five codewords of weight four and four codewords of weight six!.