Chapter 1

Preliminaries

1.1 Vector Spaces

**Definition**: A real vector space is a set $V$, whose elements are called vectors, together with two binary operations $+: V \times V \mapsto V$ and $\cdot : \mathbb{R} \times V \mapsto V$, called addition and scalar multiplication, which satisfy the following nine axioms:

(a) $u + v = v + u$ for all $u, v \in V$
(b) $(u + v) + w = u + (v + w)$ for all $u, v, w \in V$
(c) there exists $0 \in V$ such that $0 + v = v + 0$ for all $v \in V$
(d) for all $v \in V$, there exists $-v \in V$ such that $v + (-v) = 0$
(e) $(rs) \cdot u = r \cdot (s \cdot u)$ for all $u \in V$ and all $r, s \in \mathbb{R}$
(f) $(r + s) \cdot u = r \cdot u + s \cdot u$ for all $u \in V$ and all $r, s \in \mathbb{R}$
(g) $r \cdot (u + v) = r \cdot u + r \cdot v$ for all $u, v \in V$ and all $r \in \mathbb{R}$
(h) $0 \cdot u = 0$ for all $u \in V$
(i) $1 \cdot u = u$ for all $u \in V$

We normally omit the multiplication symbol $\cdot$ and use the empty notation.

**Definitions**:

1. Let $W = \{v_i \mid i \in I\}$ be a subset of $V$. We say that $W$ is linearly independent if whenever a finite linear combination $\sum_{i \in I} a_i v_i$ is zero then $a_i = 0$ for all $i \in I$. We say that $W$ is linearly dependent if there exists a finite linear combination $\sum_{i \in I} a_i v_i = 0$ with some $a_i \neq 0$.

2. A subset $W$ of $V$ spans $V$ if for each $v \in V$ there exist a finite subset $\{v_1, v_2, \ldots, v_n\}$ of $W$ and real numbers $a_1, \ldots, a_n$ such that $v = \sum_{i=1}^{n} a_i v_i$.

3. A basis of a vector space $V$ is a linearly independent spanning set.

**Theorem 1.1** Every vector space has a basis. Any two bases have the same number of elements, or all have infinitely many elements.

**Definitions**:

1. The number of elements of a basis for a vector space $V$ is called the dimension of $V$. 
(2) Let \( \beta = \{ v_i \mid i \in I \} \) be a basis for \( V \). Then every \( v \in V \) can be written uniquely as a finite sum \( v = \sum_{i \in I} a_i v_i \). The numbers \( a_i \) are called components of \( v \) with respect to \( \beta \).

(3) An inner product on a vector space \( V \) is a function \( \langle \ , \ \rangle : V \times V \to \mathbb{R} \) such that

- (a) \( \langle u, v \rangle = \langle v, u \rangle \) for all \( u, v \in V \)
- (b) \( \langle u, rv + sw \rangle = r \langle u, v \rangle + s \langle u, w \rangle \) for all \( u, v, w \in V \) and all \( r, s \in \mathbb{R} \)
- (c) \( \langle u, u \rangle \geq 0 \) and \( \langle u, u \rangle = 0 \iff u = 0 \) for all \( u \in V \)

(4) If \( V \) has an inner product then for all \( v \in V \), the length or norm of \( v \) is \( \|v\| = \sqrt{\langle v, v \rangle} \)

Remark : In \( \mathbb{R}^3 \) the ordinary dot product is defined by \( \langle (a_1, a_2, a_3), (b_1, b_2, b_3) \rangle = a_1b_1 + a_2b_2 + a_3b_3 \)

Lemma 1.2 (Cauchy-Schwarz Inequality) For all \( u, v \in V \), we have that \( |\langle u, v \rangle| \leq \|u\|\|v\| \) and we have equality if and only if \( u \) and \( v \) are linearly dependent.

Definitions : Let \( u, v \in V \).

1. If \( u \neq 0 \neq v \), then we define the angle \( \theta \) between \( u \) and \( v \) as \( \theta = \cos^{-1} \left( \frac{\langle u, v \rangle}{\|u\|\|v\|} \right) \)

2. \( u \) is orthogonal (or perpendicular) to \( v \) if \( \langle u, v \rangle = 0 \).

Theorem 1.3 If \( V \) has dimension \( n \) and an inner product, then there exists a basis \( \{ v_1, v_2, \ldots, v_n \} \) such that \( \|v_i\| = 1 \) for \( i = 1, \ldots, n \) and \( v_i \) is perpendicular to \( v_j \) for all \( 1 \leq i, j \leq n \) with \( i \neq j \).

Remark : Such a basis is called orthonormal and can be obtained by a process called the Gram-Schmidt orthogonalization.

1.2 Linear Transformations and Eigenvectors

Definitions :

1. Let \( V \) and \( W \) be vector spaces. A linear transformation from \( V \) to \( W \) is a function \( T : V \to W \) such that \( T(au + bv) = aT(u) + bT(v) \) for all \( u, v \in V \) and all \( a, b \in \mathbb{R} \).

2. A linear transformation is an isomorphism if it is one-to-one and onto.

3. Let \( T : V \to V \) be a linear transformation. A real number \( \lambda \) is an eigenvalue of \( T \) if there exists a nonzero vector \( v \in V \) such that \( T(v) = \lambda v \). Such a vector \( v \) is called an eigenvector of \( T \) corresponding to \( \lambda \).

1.3 Orientation and Cross Products

Definitions :

1. Let \( \{ u_1, u_2, \ldots, u_n \} \) and \( \{ v_1, v_2, \ldots, v_n \} \) be two ordered bases of \( V \) and define a matrix \( A = (a_{ij})_{1 \leq i, j \leq n} \) by \( v_j = \sum_{i=1}^{n} a_{ij} u_i \). We say that the two bases give the same orientation to \( V \) if \( \det(A) > 0 \) and give opposite orientations to \( V \) if \( \det(A) < 0 \).

2. We denote the basis \( \{(1,0,0), (0,1,0), (0,0,1)\} \) of \( \mathbb{R}^3 \) by \( \{e_1, e_2, e_3\} \). Its orientation will be called right handed.
(3) If \( \mathbf{u} = \sum_{i=1}^{3} a_i \mathbf{e}_i \) and \( \mathbf{v} = \sum_{i=1}^{3} b_i \mathbf{e}_i \) are vectors in \( \mathbb{R}^3 \), the cross product of \( \mathbf{u} \) and \( \mathbf{v} \) is the vector

\[
\mathbf{u} \times \mathbf{v} = (a_2 b_3 - a_3 b_2) \mathbf{e}_1 + (a_3 b_1 - a_1 b_3) \mathbf{e}_2 + (a_1 b_2 - a_2 b_1) \mathbf{e}_3
\]

By abuse of notation, this may be written as \( \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \)

**Lemma 1.4** Let \( \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3 \) and \( r \in \mathbb{R} \). Then the following holds:

(a) \( \mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u} \)

(b) \( (r \mathbf{u}) \times \mathbf{v} = r(\mathbf{u} \times \mathbf{v}) \)

(c) \( \mathbf{u} \times \mathbf{v} = \mathbf{0} \) if and only if \( \mathbf{u} \) and \( \mathbf{v} \) are linearly dependent.

(d) \( (\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w}) \)

(e) \( \mathbf{u} \times \mathbf{v} \) is perpendicular to both \( \mathbf{u} \) and \( \mathbf{v} \) under the usual dot product of \( \mathbb{R}^3 \).

(f) \( \| \mathbf{u} \times \mathbf{v} \| = \| \mathbf{u} \| \| \mathbf{v} \| \sin(\theta) \) where \( \theta \) is the angle between \( \mathbf{u} \) and \( \mathbf{v} \) (under the usual dot product)

(g) \( \{ \mathbf{u}, \mathbf{v}, \mathbf{u} \times \mathbf{v} \} \) gives a right handed orientation to \( \mathbb{R}^3 \) if \( \{ \mathbf{u}, \mathbf{v} \} \) is linearly independent.

(h) \( \langle \mathbf{u} \times \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \times \mathbf{w} \rangle \)

**Definition:** The mixed (or triple) scalar product of \( \mathbf{u}, \mathbf{v}, \mathbf{w} \) is \( [\mathbf{u}, \mathbf{v}, \mathbf{w}] = \langle \mathbf{u} \times \mathbf{v}, \mathbf{w} \rangle \)

### 1.4 Lines, Planes and Spheres

We mention the following vector equations:

(1) Let \( \mathbf{x}_0, \mathbf{v} \in \mathbb{R}^3 \) with \( \mathbf{v} \neq \mathbf{0} \). The equation of the line through \( \mathbf{x}_0 \) and parallel to \( \mathbf{v} \) is

\[
\alpha(t) = \mathbf{x}_0 + t \mathbf{v} \quad \text{where } t \in \mathbb{R}
\]

(2) Let \( \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^3 \) with \( \mathbf{x}_1 \neq \mathbf{x}_2 \). The equation of the line through \( \mathbf{x}_1 \) and \( \mathbf{x}_2 \) is

\[
\alpha(t) = \mathbf{x}_1 + t(\mathbf{x}_2 - \mathbf{x}_1) \quad \text{where } t \in \mathbb{R}
\]

(3) Let \( \mathbf{x}_0, \mathbf{n} \in \mathbb{R}^3 \) with \( \mathbf{n} \neq \mathbf{0} \). The equation of the plane through \( \mathbf{x}_0 \) and perpendicular to \( \mathbf{n} \) is

\[
\langle \mathbf{x} - \mathbf{x}_0, \mathbf{n} \rangle = 0
\]

(4) Let \( \mathbf{x}_0, \mathbf{u}, \mathbf{v} \in \mathbb{R}^3 \) with \( \{ \mathbf{u}, \mathbf{v} \} \) linearly independent. The equation of the plane through \( \mathbf{x}_0 \) and parallel to both \( \mathbf{u} \) and \( \mathbf{v} \) is

\[
\langle \mathbf{x} - \mathbf{x}_0, \mathbf{u} \times \mathbf{v} \rangle = 0
\]

(5) Let \( \mathbf{m} \in \mathbb{R}^3 \) and \( r > 0 \). The equation of the sphere with center \( \mathbf{m} \) and radius \( r \) is

\[
\langle \mathbf{x} - \mathbf{m}, \mathbf{x} - \mathbf{m} \rangle = r^2
\]
1.5 Vector Calculus

Let $V$ be a real vector space, \{v_1, v_2, \ldots, v_n\} a basis for $V$ and $f : \mathbb{R} \rightarrow V$ a function. Then we can write $f = \sum_{i=1}^{n} f_i(t)v_i$. If the component functions $f_i(t)$ are differentiable or integrable, we may differentiate or integrate $f$ componentswise:

$$\frac{df}{dt} = \sum_{i=1}^{n} \left( \frac{df_i(t)}{dt} \right) v_i \quad \text{and} \quad \int_{a}^{b} f(t) \, dt = \sum_{i=1}^{n} \left( \int_{a}^{b} f_i(t) \, dt \right) v_i$$

Note that these definitions do not depend on the choice of basis for $V$.

Similarly, if $f$ is a vector-valued function of several variables, we may take partial derivatives or multiple integrals.

**Lemma 1.5** Let $f, g : \mathbb{R} \rightarrow V$. Then the following holds:

(a) $\frac{d}{dt}(f \times g) = \frac{df}{dt} \times g + f \times \frac{dg}{dt}$

(b) Suppose that $V$ has an inner product $\langle \cdot, \cdot \rangle$. Then $\frac{d}{dt}(f, g) = \langle \frac{df}{dt}, g \rangle + \langle f, \frac{dg}{dt} \rangle$. In particular, If $\|f\|$ is constant, then $\frac{df}{dt}$ is perpendicular to $f$.

**Definitions**:

1. A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is of class $C^k$ if all derivatives up through order $k$ exist and are continuous.

2. A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is of class $C^k$ if all its (mixed) partial derivatives of order $k$ or less exist and are continuous.

3. A vector-valued function is of class $C^k$ if all of its components with respect to a given basis are of class $C^k$.

**Theorem 1.6 (chain rule)** Let $x$ be a function of the variables $u_1, u_2, \ldots, u_n$ which are functions of the variables $v_1, v_2, \ldots, v_m$. Then

$$\frac{\partial x}{\partial v_k} = \sum_{i=1}^{n} \frac{\partial x}{\partial u_i} \frac{\partial u_i}{\partial v_k} \quad \text{for} \quad k, 1, 2, \ldots, m$$
Chapter 2

Local Curve Theory

2.1 Basic Definitions

**Definition**: A regular curve in \( \mathbb{R}^3 \) is a function \( \alpha : (a, b) \mapsto \mathbb{R}^3 \) which is of class \( C^k \) for some \( k \geq 1 \) and for which \( \frac{d\alpha}{dt} \neq 0 \) for all \( t \in (a, b) \).

Let \( \alpha(t) \) be a regular curve in \( \mathbb{R}^3 \). We will mostly assume that \( \alpha(t) \) is of class \( C^3 \).

We can view a curve as the path of a particle moving in 3-space: the position at time \( t \) of the particle is given by \( \alpha(t) \).

**Definitions**:

1. A quantity is called a geometric quantity if it only depends on the image set of \( \alpha \) (in \( \mathbb{R}^3 \)) and not on the particular parametrization.

2. The **velocity vector** at \( t = t_0 \) is the derivative \( \frac{d\alpha}{dt}(t_0) \).

3. The **velocity vector field** is the vector-valued function \( \frac{d\alpha}{dt} \).

4. The **speed** at \( t = t_0 \) is \( \left\| \frac{d\alpha}{dt}(t_0) \right\| \).

5. The **tangent vector field** is the vector-valued function \( T(t) = \frac{\frac{d\alpha}{dt}}{\left\| \frac{d\alpha}{dt} \right\|} \).

6. The **tangent line** at the point \( t = t_0 \) is given by

   \[
   l = \{ w \in \mathbb{R}^3 \mid w = \alpha(t_0) + \lambda T(t_0), \lambda \in \mathbb{R} \} = \{ w \in \mathbb{R}^3 \mid w = \alpha(t_0) + \mu \frac{d\alpha}{dt}(t_0), \mu \in \mathbb{R} \}
   \]

7. A **reparametrization** of a curve \( \alpha : (a, b) \mapsto \mathbb{R}^3 \) is a bijection \( g : (c, d) \mapsto (a, b) \) such that both \( g \) and \( g^{-1} \) are of class \( C^k \) for some \( k \geq 1 \).

**Theorem 2.1** Let \( g : (c, d) \mapsto (a, b) \) be a reparametrization of the curve \( \alpha : (a, b) \mapsto \mathbb{R}^3 \). Put \( \beta = \alpha \circ g \). Then \( \beta \) is a regular curve.
Proof: Clearly $\beta$ is at least of class $C^1$. Let $r$ denote the variable on $(c, d)$. Using the chain rule, we get that

$$\frac{d\beta}{dr} = \frac{d\alpha}{dt} \frac{dg}{dr}$$

Since $\alpha$ is regular, we have that $\frac{d\alpha}{dt} \neq 0$. Note that $g(g^{-1}(t)) = t$ for all $t \in (a, b)$. Since $g$ and $g^{-1}$ are of class $C^1$, we can derive this with respect to $t$. Using the chain rule, we get that $\frac{dg}{dr} g^{-1} = 1$ for all $t \in (a, b)$. In particular, $\frac{dg}{dr} \neq 0$ for all $r \in (c, d)$. So $\frac{d\beta}{dr} \neq 0$ and $\beta$ is regular.

Remark: If $g : (c, d) \mapsto (a, b)$ is a reparametrization, then $\frac{dg}{dr} \neq 0$ for all $r \in (c, d)$. Hence we have that either $\frac{dg}{dr}$ is always positive or always negative for all $r \in (c, d)$.

Theorem 2.2 Let $\alpha : (a, b) \mapsto \mathbb{R}^3$ be a regular curve and $g : (c, d) \mapsto (a, b)$ a reparametrization. Put $\beta = \alpha \circ g$. Let $T$ (resp. $S$) be the tangent vector field of $\alpha$ (resp. $\beta$). Then $S = \pm T$.

Proof: Using the chain rule, we get that

$$S = \left| \frac{d\beta}{dr} \right| = \left| \frac{d\alpha}{dt} \right| \left| \frac{dg}{dr} \right| = \left| \frac{d\alpha}{dt} \right| \left| \frac{dg}{dr} \right| = \left| \frac{d\beta}{dr} \right| T$$

Since either $\frac{dg}{dr} > 0$ or $\frac{dg}{dr} < 0$ for all $r \in (c, d)$, we get that either $\frac{dg}{dr} = 1$ or $\frac{dg}{dr} = -1$. So $S = \pm T$.

2.2 Arc Length

Definition: Let $\alpha \mapsto \mathbb{R}^3$ be a regular curve and $[a_1, b_1] \subset (a, b)$. The length of the curve segment $\alpha : [a_1, b_1] \mapsto \mathbb{R}^3$ is

$$\int_{a_1}^{b_1} \left\| \frac{d\alpha}{dt} \right\| dt$$

Theorem 2.3 Let $\alpha : (a, b) \mapsto \mathbb{R}^3$ be a regular curve and $g : (c, d) \mapsto (a, b)$ a reparametrization of $\alpha$. Put $\beta = \alpha \circ g$. Let $[c_1, d_1] \subset [c, d]$ and $[a_1, b_1] \subset (a, b)$ such that $g([c_1, d_1]) = [a_1, b_1]$. Then the length of the curve segments $\alpha : [a_1, b_1] \mapsto \mathbb{R}^3$ and $\beta : [c_1, d_1] \mapsto \mathbb{R}^3$ are the same.

Proof: Using the chain rule, we get that the length of the curve segment $\beta : [c_1, d_1] \mapsto \mathbb{R}^3$ is

$$\int_{c_1}^{d_1} \left\| \frac{d\beta}{dr} \right\| dr = \int_{c_1}^{d_1} \left\| \frac{d\alpha}{dt} \right\| \left| \frac{dg}{dr} \right| dr = \int_{c_1}^{d_1} \left\| \frac{d\alpha}{dt} \right\| \left| \frac{dg}{dr} \right| dr$$

Suppose first that $\frac{dg}{dr} > 0$ for all $r \in (c, d)$. Then $g(c_1) = a_1$, $g(d_1) = b_1$ and $dt = \frac{dg}{dr} dr$. Hence

$$\int_{c_1}^{d_1} \left\| \frac{d\alpha}{dt} \right\| \frac{dg}{dr} dr = \int_{a_1}^{b_1} \left\| \frac{d\alpha}{dt} \right\| dt$$
which is the length of the curve segment \( \alpha : [a_1, b_1] \mapsto \mathbb{R}^3 \).

The case \( \frac{dg}{dr} < 0 \) for all \( r \in (c, d) \) is done similarly.

**Definition**: Let \( \alpha : (a, b) \mapsto \mathbb{R}^3 \) be a regular curve and \( t_0 \in (a, b) \). Put

\[
h(t) = \int_{t_0}^{t} \left\| \frac{d\alpha}{dt} \right\| dt \text{ for all } t \in (a, b)
\]

Then \( s = h(t) \) is called the *arc length along \( \alpha \).*

**Theorem 2.4** Let \( \alpha : (a, b) \mapsto \mathbb{R}^3 \) be a regular curve and \( s = h(t) \) its arc length. Then the following holds :

(a) \( \frac{ds}{dt} = \left\| \frac{d\alpha}{dt} \right\| \) for all \( t \in (a, b) \)

(b) \( s = h(t) \) is a bijection from \( (a, b) \) onto some interval \( (c, d) \) and \( h^{-1}(s) \) is a reparametrization.

(c) Put \( \beta = \alpha \circ h^{-1} \). Let \( T \) be the tangent vector field for \( \beta(s) \). Then \( T = \frac{d\beta}{ds} \).

**Proof**: (a),(b) We easily get that \( \frac{ds}{dt} = \frac{dh}{dt} = \left\| \frac{d\alpha}{dt} \right\| > 0 \) for all \( t \in (a, b) \). So \( h(t) \) is injective and hence a bijection from \( (a, b) \) onto some interval \( (c, d) \). Also, if \( \alpha \) is of class \( C^k \), then so is \( h \). Since \( h^{-1} \) is the inverse of \( h \), we get that \( \frac{dh^{-1}}{ds} = \frac{1}{\frac{dh}{ds}} \) for all \( s \in (c, d) \). Note that \( \frac{dh}{ds} \neq 0 \) for all \( t \in (a, b) \). So \( h \) and \( h^{-1} \) are both of the same class and \( h^{-1} \) is a reparametrization.

(c) Using the chain rule and (a), we get that

\[
\frac{d\beta}{ds} = \frac{d\alpha}{dt} \frac{dh^{-1}}{ds} = \frac{d\alpha}{dt} \frac{1}{\frac{dh}{ds}} = \frac{\frac{d\alpha}{dt}}{\left\| \frac{d\alpha}{dt} \right\|}
\]

In particular, \( \left\| \frac{d\beta}{ds} \right\| = 1 \) and \( T = \frac{\frac{d\beta}{ds}}{\left\| \frac{d\beta}{ds} \right\|} = \frac{d\beta}{ds} \).

**Remarks**:

(1) The variable \( s \) is the arc length.

(2) \( 0 \in (c, d) \)

(3) Practically, it may be impossible to reparametrize a curve by its arc length: we may not be able to calculate \( s = h(t) \) or \( h^{-1}(s) \).

### 2.3 Curvature and The Frenet-Serret Apparatus

**Definition**: A curve \( \alpha : (a, b) \mapsto \mathbb{R}^3 \) is a *unit speed curve* if \( \left\| \frac{d\alpha}{dt} \right\| = 1 \) for all \( t \in (a, b) \).

**Remark**: Suppose that \( \alpha : (a, b) \mapsto \mathbb{R}^3 \) is a unit speed curve. Then the arc length \( s \) is given by

\[
s = \int_{t_0}^{t} \left\| \frac{d\alpha}{dt} \right\| dt = \int_{t_0}^{t} 1 \, dt = t - t_0
\]
We will assume that $0 \in (a, b)$ and that $t_0 = 0$. Hence $s = t$ and we will write $\alpha(s)$: the curve is parametrized by its arc length. Then $T = \frac{d\alpha}{ds}$.

**Definitions**: Let $\alpha(s)$ be a unit speed curve.

1. The curvature of $\alpha(s)$ is $\kappa(s) = \|T'(s)\|

2. The principle normal vector field to $\alpha(s)$ is the vector field $N(s) = \frac{T'(s)}{\kappa(s)}$

3. The binormal vector field to $\alpha(s)$ is the vector field $B(s) = T(s) \times N(s)$

4. The torsion of $\alpha(s)$ is the real-valued function $\tau(s) = -\langle B'(s), N(s) \rangle$

5. The Frenet-Serret apparatus of $\alpha(s)$ is the set $\{\kappa(s), \tau(s), T(s), N(s), B(s)\}$

**Lemma 2.5** Let $\alpha(s)$ be a unit speed curve. Then for all $s$ such that $\kappa(s) \neq 0$, the set $\{T(s), N(s), B(s)\}$ is an orthonormal set.

**Proof**: Pick $s$ such that $\kappa(s) \neq 0$. Clearly, $\|T\| = 1$. Since $\kappa(s) = \|T'\|$, $\|N\| = 1$. By Lemma 1.5(b), $\langle T, T' \rangle = 0$ and so $\langle T, N \rangle = 0$. Let $\theta$ be the angle between $T$ and $N$. Then $\theta = 90^\circ$. Since $B = T \times N$, we have that $\langle B, T \rangle = \langle B, N \rangle = 0$ by Lemma 1.4(c) and $\|B\| = \|T\||N|\sin(\theta) = 1$ by Lemma 1.4(f).

**Proposition 2.6** Let $\alpha : (a, b) \mapsto \mathbb{R}^3$ be a unit speed curve such that $\kappa(s) = 0$ for all $s$ in $[c, d] \subset (a, b)$. Then the curve segment $\alpha : [c, d] \mapsto \mathbb{R}^3$ is a straight line.

**Proof**: Since $\|T'(s)\| = \kappa(s) = 0$ for all $s \in [c, d]$, there exists $v \in \mathbb{R}^3$ such that $T(s) = v$ for all $s \in [c, d]$. By Theorem 2.4(c), $\frac{d\alpha}{ds} = T$. Hence

$$\alpha(s) - \alpha(c) = \int_c^s \frac{d\alpha}{ds} \, ds = \int_c^s T(\sigma) \, d\sigma = \int_c^s v \, d\sigma = (s - c)v \text{ for all } s \in [c, d]$$

This is the equation of the line through $\alpha(c)$ and parallel to $v$.

**2.4 The Frenet-Serret Theorem**

**Lemma 2.7** Let $V$ be an $n$-dimensional vector space with an inner product and $E = \{e_1, e_2, \ldots, e_n\}$ an orthonormal set of $n$ elements of $V$. Then the following holds:

(a) $E$ is a basis for $V$

(b) $v = \sum_{i=1}^n \langle e_i, v \rangle e_i$ for all $v \in V$.

**Proof**: (a) Since the number of elements of $E$ is equal to the dimension of $V$, it suffices to prove that $E$ is linearly independent. Let $c_1, c_2, \ldots, c_n$ be real numbers such that $\sum_{i=1}^n c_i e_i = 0$. Then

$$0 = \left\langle \sum_{i=1}^n c_i e_i, e_j \right\rangle = \sum_{i=1}^n c_i \langle e_i, e_j \rangle = \sum_{i=1}^n c_i \delta_{ij} = c_j \text{ for } j = 1, 2, \ldots, n$$

Hence $E$ is linearly independent.

(b) Pick $v \in V$. Since $E$ is a basis for $V$, there exist real numbers $d_1, d_2, \ldots, d_n$ such that $v = \sum_{i=1}^n d_i e_i$. Hence

$$\langle e_j, v \rangle = \left\langle e_j, \sum_{i=1}^n d_i e_i \right\rangle = \sum_{i=1}^n d_i \langle e_j, e_i \rangle = \sum_{i=1}^n d_i \delta_{ij} = d_j \text{ for } j = 1, 2, \ldots, n$$

So (b) holds.
Theorem 2.8 (Frenet-Serret) Let $\alpha(s): (a,b) \mapsto \mathbb{R}^3$ be a unit speed curve with $\kappa(s) \neq 0$ for all $s \in (a,b)$ and $\{\kappa, \tau, T, N, B\}$ its Frenet-Serret apparatus. Then the following holds:

(a) $T'(s) = \kappa(s)N(s)$

(b) $N'(s) = -\kappa(s)T(s) + \tau(s)B(s)$

(c) $B'(s) = -\tau(s)N(s)$

Proof: By Lemma 2.5, $\{T(s), N(s), B(s)\}$ is an orthonormal set. By Lemma 2.7(b), we have that

$$v = \langle T, v \rangle T + \langle N, v \rangle N + \langle B, v \rangle B$$

for all $v \in \mathbb{R}^3$ (*)

(a) This follows from the definition of $\kappa$ and $T$.

(b) Differentiating $0 = \langle T, N \rangle$, we get that $0 = \langle T', N \rangle + \langle T, N' \rangle = \langle \kappa N, N \rangle + \langle T, N' \rangle = \kappa + \langle T, N' \rangle$. Hence $\langle T, N' \rangle = -\kappa$.

Since $N$ is a unit vector, $\langle N, N' \rangle = 0$ by Lemma 1.5(b).

Differentiating $0 = \langle B, N \rangle$, we get that $0 = \langle B', N \rangle + \langle B, N' \rangle = -\tau + \langle B, N \rangle$. Hence $\langle B, N' \rangle = \tau$.

Using (*) with $v = N'$, we get that

$$N' = \langle T, N' \rangle T + \langle N, N' \rangle N + \langle B, N' \rangle B = -\kappa T + \tau B$$

(c) Differentiating $0 = \langle T, B \rangle$, we get that $0 = \langle T', B \rangle + \langle T, B' \rangle = \langle \kappa N, B \rangle + \langle T, B' \rangle = \langle T, B' \rangle$. Hence $\langle T, B' \rangle = 0$.

Since $B$ is a unit vector, $\langle B, B' \rangle = 0$ by Lemma 1.5(b).

Using (*) with $v = B'$, we get that

$$B' = \langle T, B' \rangle T + \langle N, B' \rangle N + \langle B, B' \rangle B = -\tau N$$

by definition of $\tau$.

Theorem 2.9 Let $\alpha(s): (a,b) \mapsto \mathbb{R}^3$ be a unit speed curve with $\kappa(s) \neq 0$ for all $s \in (a,b)$. Then the following are equivalent:

(a) The image of $\alpha$ lies in a plane (or $\alpha$ is a plane curve).

(b) $B$ is a constant vector.

(c) $\tau(s) = 0$ for all $s \in (a,b)$.

Proof: Note that (b) and (c) are equivalent by Theorem 2.8(c).

Suppose first that (a) holds. By making an appropriate choice of coordinates in $\mathbb{R}^3$, we may assume that $\alpha$ lies in the $XY$-plane. So $\alpha(s) = (x(s), y(s), 0)$. Then we get that

$$T(s) = (x'(s), y'(s), 0) \quad \text{and} \quad T'(s) = (x''(s), y''(s), 0)$$

Since $N = \frac{T'}{\|T'\|}$, we see that both $T(s)$ and $N(s)$ are vectors in the $XY$-plane. Since $B(s)$ is a unit vector that is orthogonal to both $T(s)$ and $N(s)$, we get that $B(s) = (0, 0, \pm 1)$ for all $s \in (a,b)$. Hence $\tau(s) = -\langle B'(s), N(s) \rangle = 0$.

So (b) holds.

Suppose next that (b) holds. Let $x_0$ be an element on $\alpha$, say $x_0 = \alpha(0)$. Put $n = B(s)$ for all $s \in (a,b)$. Then we get that

$$\langle \alpha(s) - x_0, n \rangle' = \langle \alpha'(s), n \rangle = \langle T(s), B(s) \rangle = 0 \quad \text{for all} \quad s \in (a,b)$$

So $\langle \alpha(s) - x_0, n \rangle$ is a constant. Hence $\langle \alpha(s) - x_0, n \rangle = \langle \alpha(0) - x_0, n \rangle = 0$ for all $s \in (a,b)$ and $\alpha$ lies in the plane through $x_0$ and perpendicular to $n$. So (a) holds.

Definitions: Let $\alpha: (a,b) \mapsto \mathbb{R}^3$ be a unit speed curve.

1. The osculating plane of $\alpha(s)$ is the plane through $\alpha(s)$ and perpendicular to $B(s)$.
Proposition 2.10 Let \( \alpha(s) : (a, b) \mapsto \mathbb{R}^3 \) be a unit speed curve and \( x_0 \in \mathbb{R}^3 \) such that every normal plane to \( \alpha(s) \) goes through \( x_0 \) for all \( s \in (a, b) \). Then the image of \( \alpha \) lies on a sphere.

**Proof**: Since the normal plane to \( \alpha(s) \) is orthogonal to \( T(s) \), we have that \( \langle \alpha(s) - x_0, T(s) \rangle = 0 \) for all \( s \in (a, b) \). Hence \( \langle \alpha(s) - x_0, \alpha'(s) \rangle = 2 \langle \alpha(s) - x_0, \alpha''(s) \rangle = 2 \langle \alpha(s) - x_0, T(s) \rangle = 0 \) for all \( s \in (a, b) \). So there exists \( c \in \mathbb{R} \) with \( \langle \alpha(s) - x_0, \alpha(s) - x_0 \rangle = c \) for all \( s \in (a, b) \). Note that \( c \geq 0 \). If \( c = 0 \), then \( \alpha(s) = x_0 \) for all \( s \in (a, b) \) and \( \alpha \) is not regular, a contradiction. So \( c > 0 \) and the image of \( \alpha \) lies on the sphere with center \( x_0 \) and radius \( \sqrt{c} \). \( \square \)

### 2.5 The Fundamental Existence and Uniqueness Theorem for Curves

**Theorem 2.11 (Fundamental Theorem of Curves)** Let \((a, b)\) be an interval with \(a < 0 < b\), \( \bar{\kappa}(s) > 0 \) a \( C^1 \) function on \((a, b)\), \( \tilde{\tau}(s) \) a continuous function on \((a, b)\), \( x_0 \) a fixed point of \( \mathbb{R}^3 \) and \( \{D, E, F\} \) a fixed right handed orthonormal basis of \( \mathbb{R}^3 \). Then there exists a \( C^3 \) regular curve \( \alpha(s) : (a, b) \mapsto \mathbb{R}^3 \) such that

(a) The parameter \( s \) is the arc length from \( \alpha(0) \)

(b) \( \alpha(0) = x_0, \ T(0) = D, \ N(0) = E \) and \( B(0) = F \)

(c) \( \kappa(s) = \bar{\kappa}(s) \) and \( \tau(s) = \tilde{\tau}(s) \)

Moreover, \( \alpha(s) \) is unique with these properties.

**Proof**: \( \square \)

### 2.6 Non-Unit Speed Curves

Let \( \alpha(t) \) be a regular curve and \( s(t) \) its arc length. The next theorem will tell us how to determine the Frenet-Serret apparatus of \( \alpha \) in terms of \( t \). We use the prime notation for derivatives with respect to \( s \) : \( \frac{d\alpha}{ds} = \alpha', \frac{d^2\alpha}{ds^2} = \alpha'' \), etc. We denote derivatives with respect to \( t \) by dots : \( \frac{ds}{dt} = \dot{s}, \ \frac{d\alpha}{dt} = \dot{\alpha}, \ \frac{d^2\alpha}{dt^2} = \ddot{\alpha}, \) etc.

**Theorem 2.12** Let \( \alpha(t) \) be a regular curve in \( \mathbb{R}^3 \). Then the following holds:

(a) \( T = \frac{\dot{\alpha}}{\|\dot{\alpha}\|} \)

(b) \( B = \frac{\dot{\alpha} \times \ddot{\alpha}}{\|\dot{\alpha} \times \ddot{\alpha}\|} \)

(c) \( N = B \times T \)

(d) \( \kappa = \frac{\|\dot{\alpha} \times \ddot{\alpha}\|}{\|\dot{\alpha}\|^3} \)

(e) \( \tau = \left[ \frac{\dot{\alpha}, \ddot{\alpha}, \dot{\alpha}}{\|\dot{\alpha} \times \ddot{\alpha}\|^2} \right] \)
Proof : Using the chain rule, we get that
\[ \dot{\alpha} = \alpha' \dot{s} = s T \quad (*) \]

Note that \( \dot{s} = \| \dot{\alpha} \| \). So (a) follows.

Deriving (*) with respect to \( t \) and using the chain rule and the Frenet-Serret equations, we get that
\[ \ddot{\alpha} = \ddot{s} T + \dot{s} \ddot{T} = \ddot{s} T + \dot{s} \dddot{T} + \dot{s}^2 \kappa N \quad (**) \]

Taking the cross product of (*) with (**), we find that
\[ \dot{\alpha} \times \ddot{\alpha} = (\dot{s} T) \times (\ddot{s} T + \dot{s}^2 \kappa N) = \dot{s}^3 \kappa T \times N = \dot{s}^3 \kappa B \quad (***) \]

Taking the norm on both sides of (***) , we get that
\[ \| \dot{\alpha} \times \ddot{\alpha} \| = \| \dot{s}^3 \kappa B \| = | \dot{s}^3 \kappa | \| B \| = \dot{s}^3 \kappa = \| \dot{\alpha} \| \| \ddot{\alpha} \| \]

which proves (d).

Suppose that \( \kappa \neq 0 \). Then (b) follows immediately from (**). Since \( \{ T, N, B \} \) is a right handed orthonormal basis, (c) holds.

Deriving (**) with respect to \( t \) and using the chain rule and the Frenet Serret equations, we get that
\[ \dddot{\alpha} = \dddot{s} T + \ddot{s} \dddot{T} + (\dot{s}^2 \kappa) N + \dot{s}^2 \kappa N' = \dddot{s} T + \ddot{s} \dddot{T} + (\dot{s}^2 \kappa) N + \dot{s}^2 \kappa N' + \dddot{s} \kappa N + (\dot{s}^2 \kappa) N - \dot{s}^3 \kappa^2 T + \dot{s}^3 \kappa T B \]

Hence \( \dddot{\alpha} = (\dddot{s} - \dot{s}^3 \kappa^2) T + (\dddot{s} \kappa + (\dot{s}^2 \kappa^2)) N + \dot{s}^3 \kappa T B \) and
\[ [\alpha, \dddot{\alpha}, \dddot{\alpha}] = \langle \dot{\alpha} \times \dddot{\alpha}, \dddot{\alpha} \rangle = \langle \dot{s}^3 \kappa B, \dddot{\alpha} \rangle = \langle \dot{s}^3 \kappa B, \dot{s}^3 \kappa T B \rangle = \dot{s}^6 \kappa^2 \tau = \tau (\dot{s}^3 \kappa)^2 = \tau \| \dot{\alpha} \times \dddot{\alpha} \|^2 \]

which proves (e). \( \Box \)

Theorem 2.13 Let \( \alpha(t) \) be a regular curve in \( \mathbb{R}^3 \). Then the following holds :

(a) \( \dot{T} = \kappa \| \dot{\alpha} \| N \)
(b) \( \dot{N} = -\kappa \| \dot{\alpha} \| T + \tau \| \dot{\alpha} \| B \)
(c) \( \dot{B} = -\tau \| \dot{\alpha} \| N \)

Proof : (a) Using the chain rule and the Frenet-Serret equations, we get that \( \dot{T} = T' \dot{s} = \kappa N \| \dot{\alpha} \| \), which proves (a).

(b) and (c) are done similarly. \( \Box \)
Chapter 3
Global Theory of Plane Curves

In this chapter, we will study plane curves (so $\tau \equiv 0$). We will assume that a suitable choice of coordinates has been made in $\mathbb{R}^3$ so that the curve lies in the $XY$-plane. We will disregard the Z-coordinate and write the curve as if it lies in $\mathbb{R}^2$.

3.1 The Rotation Index of a Plane Curve

Definitions: Let $\alpha(s): (a,b) \rightarrow \mathbb{R}^2$ be a unit speed $C^2$ plane curve.

(a) The tangent vector field $t(s)$ to $\alpha$ is $t(s) = \alpha'(s)$ for all $s \in (a,b)$.

(b) The normal vector field $n(s)$ to $\alpha$ is the unique unit vector field $n(s)$ such that $\{t(s), n(s)\}$ gives a right handed orthonormal basis of $\mathbb{R}^2$ for each $s \in (a,b)$.

(c) The plane curvature $k(s)$ of $\alpha$ is given by $k(s) = \langle t'(s), n(s) \rangle$

Theorem 3.1 Let $\alpha(s): (a,b) \rightarrow \mathbb{R}^2$ be a unit speed plane curve. Then the following holds:

(a) $t'(s) = k(s)n(s)$ for all $s \in (a,b)$.

(b) If $\alpha(s) = (x(s), y(s))$ for all $s \in (a,b)$ and $x, y$ are of class $C^2$, then

$$t(s) = (x'(s), y'(s)), \quad n(s) = (-y'(s), x'(s))$$

and $k(s) = x'(s)y''(s) - y'(s)x''(s)$ for all $s \in (a,b)$

(c) $t(s) = T(s)$ for all $s \in (a,b)$

(d) Suppose that $N(s)$ exists for all $s \in (a,b)$. Then $n(s) = \pm N(s)$, $k(s) = |k(s)|$, $n(s)$ is differentiable and $n'(s) = -k(s)t(s)$ for all $s \in (a,b)$.

Proof: (a) Since $t(s)$ is a unit vector, we have that $\langle t(s), t'(s) \rangle = 0$ for all $s \in (a,b)$ by Lemma 1.5(b). Hence $t'(s) = (t'(s), t(s))t(s) + (t'(s), n(s))n(s) = k(s)n(s)$ by Lemma 2.7(b).

(b) Clearly, $t(s) = (x'(s), y'(s))$. Put $m(s) = (-y'(s), x'(s))$ for all $s \in (a,b)$. Pick $s \in (a,b)$. Then $\langle t(s), m(s) \rangle = 0$ and $\|m(s)\| = \sqrt{(-y'(s))^2 + (x'(s))^2} = \|t(s)\| = 1$. Hence $\{t(s), m(s)\}$ is an orthonormal basis for $\mathbb{R}^2$. The change of basis matrix $A$ to change from the standard basis $\{(1,0), (0,1)\}$ to $\{t(s), m(s)\}$ is

$$A = \begin{bmatrix} x'(s) & -y'(s) \\ y'(s) & x'(s) \end{bmatrix}.$$ 

Since det($A$) = $(x'(s))^2 + (y'(s))^2 = \|t(s)\|^2 = 1$, $\{t(s), m(s)\}$ is a right handed basis. So $m(s) = n(s)$.

Finally, $k(s) = \langle t'(s), n(s) \rangle = \langle (x''(s), y''(s)), (-y'(s), x'(s)) \rangle = x'(s)y''(s) - y'(s)x''(s)$.

(c) This is obvious.

(d) Suppose that $N(s)$ exists for all $s \in (a,b)$. Since $N(s)$ is a unit vector in the $XY$-plane, orthogonal to $T(s) = t(s)$, we have that $n(s) = \pm N(s)$. So $(x'', y'') = T' = \kappa N = \pm \kappa n = \pm \kappa(-y', x')$. Suppose that $(x'', y'') = \kappa(-y', x')$. Then $k = x'y'' - y'x'' = \kappa(x')^2 + \kappa(y')^2 = \kappa \|t\|^2 = \kappa$ and $n' = (-y'', x'') = (-\kappa x', -\kappa y') = -kt$. Similarly for $(x'', y'') = -\kappa(-y', x')$. \qed
Theorem 3.2 Let \( \alpha(t) : (a, b) \to \mathbb{R}^2 : t \mapsto (x(t), y(t)) \) be a regular plane curve. Then

\[
\mathbf{t} = \frac{\mathbf{\alpha}'}{\|\mathbf{\alpha}'\|} = \frac{(\dot{x}, \dot{y})}{\sqrt{\dot{x}^2 + \dot{y}^2}}, \quad \mathbf{n} = \frac{(-\dot{y}, \dot{x})}{\sqrt{\dot{x}^2 + \dot{y}^2}} \quad \text{and} \quad k = \frac{\ddot{x} \dot{y} - \ddot{y} \dot{x}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}
\]

Proof : Let \( s(t) \) be the arc length along \( \alpha \). Then \( \dot{s} = \|\mathbf{\alpha}'\| \). We use the prime notation for derivatives with respect to \( s \). Using the chain rule, we get that \( \mathbf{\alpha} = \mathbf{\alpha}' \dot{s} = t \dot{s} \). Hence \( \mathbf{t} = \frac{\mathbf{\alpha}'}{\|\mathbf{\alpha}'\|} = \frac{(\dot{x}, \dot{y})}{\sqrt{\dot{x}^2 + \dot{y}^2}} \). By Theorem 3.1(b),

\[
\mathbf{n} = \frac{(-\dot{y}, \dot{x})}{\sqrt{\dot{x}^2 + \dot{y}^2}}
\]

Using the chain rule and Theorem 3.1(a), we get that \( \mathbf{\alpha}' = \mathbf{t} \dot{s} + s \ddot{s} = \mathbf{t} \dot{s}^2 + s \ddot{s} = k \dot{s}^2 \mathbf{n} + s \ddot{s} \). So

\[
\mathbf{\alpha} \times \mathbf{\alpha}' = (s \dot{t}) \times (k \dot{s}^2 \mathbf{n} + s \ddot{s}) = k \dot{s}^3 \mathbf{t} \times \mathbf{n} = k \dot{s}^3 \mathbf{e}_3
\]

since \( \{\mathbf{t}, \mathbf{n}\} \) is a right handed orthonormal basis of \( \mathbb{R}^2 \). But \( \mathbf{\alpha}' = (\dot{x}, \dot{y}) \) and \( \mathbf{\alpha} = (\dot{x}, \dot{y}) \) and so

\[
\mathbf{\alpha} \times \mathbf{\alpha}' = \begin{vmatrix}
\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\
\dot{x} & \dot{y} & 0 \\
\dot{y} & -\dot{x} & 0
\end{vmatrix} = (\dot{x} \dot{y} - \dot{y} \dot{x}) \mathbf{e}_3
\]

Hence \( k \dot{s}^3 = \dot{x} \dot{y} - \dot{y} \dot{x} \) and so \( k = \frac{\dot{x} \dot{y} - \dot{y} \dot{x}}{s^3} = \frac{\dot{x} \dot{y} - \dot{y} \dot{x}}{(\dot{x}^2 + \dot{y}^2)^{3/2}} \). \( \square \)

Definitions :

(a) A function \( f : \mathbb{R} \to V \) is periodic if there exists a constant \( a > 0 \) such that \( f(t) = f(a + t) \) for all \( t \in \mathbb{R} \). The period of \( f \) is the smallest such number \( a \).

(b) A regular curve \( \alpha : \mathbb{R} \to \mathbb{R}^3 \) is closed if \( \alpha \) is periodic.

(c) A regular curve \( \alpha(t) \) is simple if one of the following holds:
   
   (1) \( \alpha(t) \) is one-to-one

   (2) \( \alpha(t) \) is a closed curve with period \( a \) such that \( \alpha(t_1) \neq \alpha(t_2) \) for all \( t_1, t_2 \in [0, a] \) with \( t_1 \neq t_2 \)

(d) Let \( \alpha(s) \) be a regular closed unit speed plane curve with period \( L \). We define \( \theta(s) = \int_0^s k(\sigma) \, d\sigma \) for all \( s \in [0, L] \).

Lemma 3.3 Let \( \alpha(s) \) be a regular closed unit speed plane curve with period \( L \) and \( \theta(s) = \int_0^s k(\sigma) \, d\sigma \) for all \( s \in [0, L] \). If \( \mathbf{t}(0) = (1, 0) \) then \( \mathbf{t}(s) = (\cos(\theta(s)), \sin(\theta(s))) \) for all \( s \in [0, L] \).

Proof : Define a continuous function \( \varphi(s) \) by \( \mathbf{t}(s) = (\cos(\varphi(s)), \sin(\varphi(s))) \) for all \( s \in [0, L] \). Since \( \mathbf{t}(0) = (1, 0) \), we can choose \( \varphi(0) = 0 \). By Lemma 3.1(b), we get that

\[
\mathbf{t}'(s) = \varphi'(s) (-\sin(\varphi(s)), \cos(\varphi(s))) = \varphi'(s) \mathbf{n}(s) = k(s) \mathbf{n}(s) \quad \text{for all} \quad s \in [0, L]
\]

Hence \( \varphi'(s) = k(s) = \theta'(s) \) for all \( s \in [0, L] \). So there exists a constant \( c \) with \( \theta(s) = \varphi(s) + c \) for all \( s \in [0, L] \). Since \( \theta(0) = 0 = \varphi(0) \), we have that \( \theta(s) = \varphi(s) \) and \( \mathbf{t}(s) = (\cos(\theta(s)), \sin(\theta(s))) \) for all \( s \in [0, L] \). \( \square \)

Definition : Let \( \alpha(s) \) be a regular closed unit speed plane curve with period \( L \). The rotation index of \( \alpha(s) \) is the integer

\[
i \alpha = \frac{\theta(L) - \theta(0)}{2\pi} = \frac{\theta(L)}{2\pi}
\]

Theorem 3.4 (Rotation Index Theorem) The rotation index of a regular simple closed unit speed plane curve is \( \pm 1 \).

Proof :
Definitions:

(a) A **piecewise C^k regular curve** is a continuous function \( \alpha(s) : [a, b] \to \mathbb{R}^3 \) and a finite set of points \( a = s_0 < s_1 < \cdots < s_{n-1} < s_n = b \) such that \( \alpha : (s_i, s_{i+1}) \to \mathbb{R}^3 \) is a regular \( C^k \) curve for \( i = 0, 1, \ldots, n - 1 \).

Suppose that \( \alpha(s) : [a, b] \to \mathbb{R}^2 \) is a piecewise regular plane curve. Let \( \delta \theta_i \) be the angle through which \( t \) rotates on the segment \( \alpha : (s_i, s_{i+1}) \to \mathbb{R}^2 \) for \( i = 0, 1, \ldots, n - 1 \). Put \( t^-_i(s) \) = \( \lim_{s \to s_i^-} t(s) \) for \( i = 1, 2, \ldots, n \) and \( t^+_i(s) \) = \( \lim_{s \to s_i^+} t(s) \) for \( i = 0, 1, \ldots, n - 1 \). Let \( \Delta \theta_i \in [-\pi, \pi] \) be the angle between \( t^-_i(s) \) and \( t^+_i(s) \) (measured form \( t^-_i(s) \) to \( t^+_i(s) \)) for \( i = 1, 2, \ldots, n - 1 \) (if \( \alpha \) is closed, put \( \Delta \theta_0 = \Delta \theta_n \) the angle between \( t^-_n(s) \) and \( t^+_n(s) \)). If \( |\Delta \theta_i| = \pi \), we can figure out the sign of \( \Delta \theta_i \) by finding the sign of the angle between \( t(s_i - \varepsilon) \) and \( t(s_i + \varepsilon) \) for some small \( \varepsilon \).

(b) The **rotation index** of a piecewise regular closed plane curve \( \alpha \) is the integer

\[
i_\alpha = \sum_{i=0}^{n-1} \left( \delta \theta_i + \Delta \theta_i \right) \frac{1}{2\pi}
\]

**Theorem 3.5** The rotation index of a piecewise regular simple closed plane curve is \( \pm 1 \).

\[\square\]

3.2 The Isoperimetric Inequality

**Theorem 3.6 (Green)** Let \( C \) be a closed plane \( C^2 \) curve, which bounds a region \( R \) and is traversed counterclockwise. Then

\[
\int_C (f \, dx + g \, dy) = \iint_R \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \, dx \, dy
\]

for all differentiable functions \( f \) and \( g \) defined on \( R \).

\[\square\]

**Lemma 3.7** Let \( \alpha(t) : [a, b] \to \mathbb{R}^2 \) be a simple closed plane curve which is traversed counterclockwise and whose image bounds a region \( R \). Then the area of \( R \) is given by

\[
\int_a^b x(t)y'(t) \, dt = -\int_a^b y(t)x'(t) \, dt
\]

\[\square\]

**Proof:** By definition, the area of \( R \) is given by \( \int \int_R 1 \, dx \, dy \). Using Green’s Theorem with \( f(x, y) = 0 \) and \( g(x, y) = x \), we get that

\[
\int x \, dy = \int \alpha (f \, dx + g \, dy) = \iint_R \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \, dx \, dy = \int \int_R 1 \, dx \, dy = \text{area of } R
\]

Since \( \alpha(t) = (x(t), y(t)) \) for all \( t \in [a, b] \), we get that

\[
\int_a^b x \, dy = \int_a^b x(t)y'(t) \, dt = \int_a^b x(t)y'(t) \, dt
\]

Similarly, by using Green’s Theorem with \( f(x, y) = -y \) and \( g(x, y) = 0 \) we get that the area of \( R \) is given by

\[\square\]

**Lemma 3.8** Let \( a \) and \( b \) be positive real numbers. Then \( \sqrt{ab} \leq \frac{a+b}{2} \). Moreover, we have equality if and only if \( a = b \).

\[\square\]

**Proof:** We have that

\[
\sqrt{ab} \leq \frac{a+b}{2} \iff 2\sqrt{ab} \leq a+b \iff 4ab = (2\sqrt{ab})^2 \leq (a+b)^2 = a^2 + 2ab + b^2 \iff 0 \leq a^2 - 2ab + b^2 \iff 0 \leq (a-b)^2
\]

This proves the lemma.
\textbf{Theorem 3.9 (Isoperimetric Inequality)} Let $\alpha$ be a simple closed regular plane curve of length (perimeter) $L$. Let $A$ be the area of the region bounded by $\alpha$. Then $L^2 \geq 4\pi A$. Moreover, we have equality if and only if $\alpha$ is a circle.

\textbf{Proof :} Let $l_1, l_2$ be two parallel lines tangent to $\alpha$ with $\alpha$ bounded between them. Let $\beta$ be a circle tangent to $l_1$ and $l_2$ which does not intersect $\alpha$. Let $r$ be the radius of that circle (note that for now, $r$ depends on $l_1$). We choose a coordinate system with the origin at the center of the circle and the $Y$-axis parallel to $l_1$. Let $s \in [0, L]$ be the arc length on $\alpha$. Put $\alpha(s) = (x(s), y(s))$. Then $\beta(s) = (x(s), \gamma(s))$. Note that $s$ is not the arc length on $\beta$. By Lemma 3.7, the area of the region bounded by $\alpha$ is $A = \int_0^L x(s)y'(s) \, ds$ and the area of the region bounded by $\beta$ is $\pi r^2 = -\int_0^L \gamma(s)x'(s) \, ds$. Hence

$$A + \pi r^2 = \int_0^L (x(s)y'(s) - \gamma(s)x'(s)) \, ds = \int_0^L ((x', y'), (-\gamma, x)) \, ds \leq \int_0^L |(x', y'), (-\gamma, x)| \, ds$$

By the Cauchy-Schwarz Inequality (Lemma 1.2), we get that

$$|(x', y'), (-\gamma, x)| \leq \| (x', y')\| \|(-\gamma, x)\| = \sqrt{(x')^2 + (y')^2} \sqrt{(-\gamma)^2 + x^2} = \|\alpha'(s)\|\|\beta(s)\| = 1 \cdot r = r$$

Hence we have that

$$A + \pi r^2 = \int_0^L ((x', y'), (-\gamma, x)) \, ds \leq \int_0^L |(x', y'), (-\gamma, x)| \, ds \leq \int_0^L r \, ds = rL \quad (*)$$

By Lemma 3.8 (with $a = A$ and $b = \pi r^2$), we get that

$$\sqrt{A + \pi r^2} \leq \frac{A + \pi r^2}{2} \leq \frac{rL}{2} \quad (**)$$

Hence $A \pi r^2 \leq \frac{r^2L^2}{4}$ and so $L^2 \geq 4\pi A$.

Suppose now that $L^2 = 4\pi A$. So we have equalities in (**). By Lemma 3.8, $A = \pi r^2$. Moreover, we have equalities in (**). So $|(x', y'), (-\gamma, x)| = r$. By Lemma 1.2, $(-\gamma, x)$ and $(x', y')$ are linearly dependent. So $(-\gamma, x) = c(x', y')$ for some $c \in \mathbb{R}$. Then $r = |(x', y'), (-\gamma, x)| = |(x', y'), c(x', y')| = c((x')^2 + (y')^2) = c\|\alpha'\|^2 = c$. Hence we have that $x(s) = ry'(s)$ for all $s \in (0, L)$.

Since $L = \pi r^2$, we see that $r$ only depends on $A$, not on $l_1$. So we can go over the same construction and calculations with lines $l_3$ and $l_4$ which are perpendicular to $l_1$. Denote the coordinates in the new coordinate system (related to $l_3$ and $l_4$) by $(\tilde{x}, \tilde{y})$. Then we have again that $\tilde{x}(s) = ry'(s)$ for all $s \in (0, L)$. Note that the $\tilde{x}$-axis (resp. $\tilde{y}$-axis) is parallel to and pointing in the same (resp. opposite) direction of the $Y$-axis (resp. $x$-axis). Hence there are constants $a$ and $b$ such that $\tilde{x}(s) = y(s) - b$ and $\tilde{y}(s) = a - x(s)$ for all $s \in (0, L)$. So $y(s) - b = \tilde{x}(s) = ry'(s) = r(a - x(s))' = -rx'(s)$ for all $s \in (0, L)$. Hence

$$x^2(s) + (y(s) - b)^2 = (ry'(s))^2 + (-rx'(s))^2 = r^2((x'(s)^2 + (y'(s))^2) = r^2\|\alpha'(s)\|^2 = r^2 \quad \text{for all } s \in (0, L)$$

So $\alpha$ is a circle with radius $r$ and center $(0, b)$ in the $xy$-coordinate system. $\square$

\subsection{3.3 Convex Curves and The Four-Vertex Theorem}

\textbf{Definition :} A regular curve is \textit{convex} if it lies on one side of each tangent line.

\textbf{Theorem 3.10} A simple closed regular plane curve $\alpha(s)$ is convex if and only if $k(s)$ has constant sign.

\textbf{Proof :}
Remark: If \( \alpha \) is a regular plane curve, \( P \) is a point on \( \alpha \) and \( L \) is a line through \( P \) such that \( \alpha \) lies on one side of \( L \), then \( L \) is the tangent line to \( \alpha \) at \( P \).

Definition: A vertex of a regular plane curve is a point where the plane curvature \( k \) has a relative minimum or maximum.

Lemma 3.11 Let \( \alpha(s) : [0, L] \rightarrow \mathbb{R}^2 : s \mapsto (x(s), y(s)) \) be a regular unit speed closed plane curve and \( A, B, C \in \mathbb{R} \). Then
\[
\int_0^L (Ax(s) + By(s) + C)k'(s) \, ds = 0.
\]

Proof: By Theorem 3.1(a)(b), we have that \( L \) is not identically zero, a contradiction to Lemma 3.11. \( \alpha \) is not identically zero on \((0, \infty)\). Suppose that \( \alpha \) is not identically zero on \((0, \infty)\). Then there exists \( \alpha \) such that tangent line, a contradiction since \( \alpha \) is convex. So the tangent line to \( \alpha \) is parallel to \( \alpha \) at \( q \) and \( r \) are on different sides of that tangent line, a contradiction since \( \alpha \) is convex. So \( \alpha \) is a line segment around the point \( p \). But then \( k(s) \) is constant on some interval, a contradiction. Assume next that \( L_2 \neq L_1 \). Since \( \alpha \) is convex, \( \alpha \) lies between the parallel lines \( L_1 \) and \( L_2 \), a contradiction since \( \alpha \) is simple and closed.

So the tangent line to \( \alpha \) at \( p \) is not the same as the tangent line to \( \alpha \) at \( q \). Let \( L \) be the line through \( p \) and \( q \) and \( \beta \) (resp. \( \gamma \)) be the ‘arc’ of \( \alpha \) on \([0, s_1]\) (resp. \([s_1, L]\)). Suppose \( \beta \) does not lie on one side of \( L \). Then the curve \( \alpha \) meets \( L \) at a point \( r \) with \( p \neq r \neq q \). Consider the intermediate point of \( p \) and \( r \), say \( y \). If \( L \neq \alpha \) is not the tangent line to \( \alpha \) at \( p \), then \( q \) and \( r \) are on different sides of that tangent line, a contradiction since \( \alpha \) is convex. So \( L \) is the tangent line to \( \alpha \) at \( p \). Then \( L \) is a line that meets \( \alpha \) in \( q \) (resp. \( r \)) such that \( \alpha \) lies on one side of \( L \). Hence \( L \) is also the tangent line to \( \alpha \) at \( q \) (resp. \( r \)), a contradiction since \( \alpha \) and \( \gamma \) have different tangent lines.

So \( \beta \) lies on one side of \( L \). Similarly, we get that \( \gamma \) lies on one side of \( L \). Suppose that \( \beta \) and \( \gamma \) lie on the same side of \( L \). Then \( L \) is a line going through \( p \) (resp. \( q \)) such that \( \alpha \) lies on one side of \( L \). Hence \( L \) is the tangent line to \( \alpha \) at \( p \) and \( q \), again a contradiction since \( \alpha \) and \( \beta \) have different tangent lines.

So \( \beta \) and \( \gamma \) lie on different sides of \( L \). Let \( Ax + By + C = 0 \) be the equation of \( L \). Then we may assume that \( Ax(s) + By(s) + C \geq 0 \) for all \( s \in (0, s_1) \) and \( Ax(s) + By(s) + C \leq 0 \) for all \( s \in (s_1, L) \). Moreover, \( Ax(s) + By(s) + C \) is not identically zero on \((0, L)\) (otherwise \( \alpha \) contains a line segment and \( k(s) \) would be constant on some interval). Suppose that \( \alpha \) has only two vertices. We may assume that \( k \) reaches a minimum at \( p \) and a maximum at \( q \). Then \( k'(s) \geq 0 \) for all \( s \in (0, s_1) \) and \( k'(s) \leq 0 \) for all \( s \in (s_1, L) \). Hence \( (Ax(s) + By(s) + C)k'(s) \geq 0 \) for all \( s \in (0, L) \) and is not identically zero, a contradiction to Lemma 3.11. So \( \alpha \) has a third vertex, say on \( L \). Hence \( k'(s) \) changes signs on \( L \), say at \( s_2 \) with \( 0 < s_2 < s_1 \). Then \( k'(s) > 0 \) for \( 0 < s < s_2 \), \( k'(s) < 0 \) for \( s_2 < s < s_1 \) and \( k'(s) > 0 \) for \( s_1 < s < L \), a contradiction since \( k'(s) \) has a minimum at \( s = 0 \).

Hence \( \alpha \) has at least four vertices. \( \square \)
Chapter 4

Local Surface Theory

4.1 Basic Definitions

Definitions

(a) A subset \( \mathcal{U} \) of \( \mathbb{R}^2 \) is open if for every point \((a, b) \in \mathcal{U}\), there exists some real number \( \varepsilon > 0 \) such that \((x, y) \in \mathcal{U}\) whenever \((x-a)^2 + (y-b)^2 < \varepsilon^2\).

(b) Let \( \mathcal{U} \) be an open subset \( \mathbb{R}^2 \) (with variables \((u_1, u_2)\)) and \( \mathbf{r} : \mathcal{U} \rightarrow \mathbb{R}^3 \) a function. Then \( \mathbf{r} \) is a \( C^k \) coordinate patch (or simple surface) if \( \mathbf{r} \) is one-to-one, of class \( C^k \) and \( \frac{\partial \mathbf{r}}{\partial u_1} \times \frac{\partial \mathbf{r}}{\partial u_2} \neq 0 \) on \( \mathcal{U} \).

(c) Let \( \mathcal{U} \) and \( \mathcal{V} \) be open subsets of \( \mathbb{R}^2 \) (with variables \((v_1, v_2)\) on \( \mathcal{V} \) and variables \((u_1, u_2)\) on \( \mathcal{U} \)) and \( f : \mathcal{V} \rightarrow \mathcal{U} : (v_1, v_2) \mapsto (f_1(v_1, v_2), f_2(v_1, v_2)) \) a function. Then the Jacobian of \( f \) (notation : \( J(f) \)) is the matrix

\[
J(f) = \begin{bmatrix}
\frac{\partial f_1}{\partial v_1} & \frac{\partial f_1}{\partial v_2} \\
\frac{\partial f_2}{\partial v_1} & \frac{\partial f_2}{\partial v_2}
\end{bmatrix}
\]

(d) Let \( \mathcal{U} \) and \( \mathcal{V} \) be open subsets of \( \mathbb{R}^2 \) and \( f : \mathcal{V} \rightarrow \mathcal{U} \) a function. Then \( f \) is a \( C^k \) coordinate transformation if \( f \) is a bijection of class \( C^k \) and \( f^{-1} \) is also of class \( C^k \).

Theorem 4.1 Let \( \mathcal{U} \) and \( \mathcal{V} \) be open subsets of \( \mathbb{R}^2 \) and \( f : \mathcal{V} \rightarrow \mathcal{U} \) a bijection of class \( C^k \). Then \( f \) is a coordinate transformation if and only if \( \det(J(f)) \neq 0 \) in \( \mathcal{V} \).

Proof: Suppose first that \( \det(J(f)) \neq 0 \) on \( \mathcal{V} \). Then by the Inverse Function Theorem (see any advanced calculus course), \( f^{-1} \) is of class \( C^k \). So \( f \) is a coordinate transformation.

Suppose next that \( f \) is a coordinate transformation. Let \((v_1, v_2)\) be the coordinates on \( \mathcal{V} \) and \((u_1, u_2)\) the coordinates on \( \mathcal{U} \). Let \( g : \mathcal{U} \rightarrow \mathcal{V} : (u_1, u_2) \mapsto (g_1(u_1, u_2), g_2(u_1, u_2)) \) be the inverse of \( f \). Then \( f \circ g \) is the identity function on \( \mathcal{U} \).

So

\[
f_1(g_1(u_1, u_2), g_2(u_1, u_2)) = u_1 \quad \text{and} \quad f_2(g_1(u_1, u_2), g_2(u_1, u_2)) = u_2 \quad \text{for all} \ (u_1, u_2) \in \mathcal{U}
\]

We use the chain rule to derive these equations with respect to \( u_1 \) and \( u_2 \):

\[
\frac{\partial f_1}{\partial v_1} \frac{\partial g_1}{\partial u_1} + \frac{\partial f_1}{\partial v_2} \frac{\partial g_2}{\partial u_1} = 1 \quad , \quad \frac{\partial f_1}{\partial v_1} \frac{\partial g_1}{\partial u_2} + \frac{\partial f_1}{\partial v_2} \frac{\partial g_2}{\partial u_2} = 0
\]

\[
\frac{\partial f_2}{\partial v_1} \frac{\partial g_1}{\partial u_1} + \frac{\partial f_2}{\partial v_2} \frac{\partial g_2}{\partial u_1} = 0 \quad , \quad \frac{\partial f_2}{\partial v_1} \frac{\partial g_1}{\partial u_2} + \frac{\partial f_2}{\partial v_2} \frac{\partial g_2}{\partial u_2} = 1
\]
We can rewrite these equations in matrix form:

\[
\begin{bmatrix}
\frac{\partial f_1}{\partial v_1} & \frac{\partial f_1}{\partial v_2} \\
\frac{\partial f_2}{\partial v_1} & \frac{\partial f_2}{\partial v_2}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial g_1}{\partial u_1} & \frac{\partial g_1}{\partial u_2} \\
\frac{\partial g_2}{\partial u_1} & \frac{\partial g_2}{\partial u_2}
\end{bmatrix}
= \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

or \(J(f)J(g) = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}\)

So the matrix \(J(f)\) has an inverse. Hence \(\det(J(f)) \neq 0\) for all \((v_1, v_2) \in \mathcal{V}\). \(\square\)

**Theorem 4.2** Let \(\mathcal{U}\) and \(\mathcal{V}\) be open subsets of \(\mathbb{R}^2\), \(g: \mathcal{U} \mapsto \mathbb{R}\) a continuous function and \(f: \mathcal{V} \mapsto \mathcal{U}\) a \(C^1\) coordinate transformation. Then

\[
\int\int_{\mathcal{U}} g(u_1, u_2) \, du_1 \, du_2 = \int\int_{\mathcal{V}} g(f_1(v_1, v_2), f_2(v_1, v_2)) \, |\det(J(f))| \, dv_1 \, dv_2
\]

**Proof**

\(\square\)

**Lemma 4.3** Let \(r: \mathcal{U} \mapsto \mathbb{R}^3\) be a simple surface of class \(C^k\) and \(f: \mathcal{V} \mapsto \mathcal{U}\) a \(C^k\) coordinate transformation. Then \(r \circ f: \mathcal{V} \mapsto \mathbb{R}^3\) is a simple surface of class \(C^k\).

**Proof**

Note that \(r \circ f\) is one-to-one and of class \(C^k\). Also, \(r(v_1, v_2) = r(f_1(v_1, v_2), f_2(v_1, v_2))\). Using the chain rule, we get that

\[
\frac{\partial r}{\partial v_1} = \left(\frac{\partial r}{\partial u_1} \cdot \frac{\partial f_1}{\partial v_1} + \frac{\partial r}{\partial u_2} \cdot \frac{\partial f_2}{\partial v_1}\right)
\]

Recall that \(u \times u = 0\) and \(u \times v = -v \times u\) for all \(u, v \in \mathbb{R}^3\). Hence we get that

\[
\frac{\partial r}{\partial v_1} = \left(\frac{\partial f_1}{\partial v_1} \cdot \frac{\partial r}{\partial u_1} + \frac{\partial f_2}{\partial v_1} \cdot \frac{\partial r}{\partial u_2}\right)
\]

Since \(r(v_1, v_2)\) is a simple surface, we have that \(\frac{\partial r}{\partial u_1} \times \frac{\partial r}{\partial u_2} \neq 0\). Since \(f\) is a coordinate transformation, \(\det(J(f)) \neq 0\) by Theorem 4.1. Hence \(\frac{\partial r}{\partial v_1} \times \frac{\partial r}{\partial v_2} \neq 0\) and so \(r(v_1, v_2)\) is a regular surface. \(\square\)

**IMPORTANT NOTATION**

If \(r(u_1, u_2)\) is a simple surface, we put \(r_1(a, b) = \frac{\partial r}{\partial u_1}(a, b)\) and \(r_2(a, b) = \frac{\partial r}{\partial u_2}(a, b)\).

Similarly, if \(r(u, v)\) is a simple surface, we put \(r_u(a, b) = \frac{\partial r}{\partial u}(a, b)\) and \(r_v(a, b) = \frac{\partial r}{\partial v}(a, b)\).

**Definitions**

Let \(r: \mathcal{U} \mapsto \mathbb{R}^3\) be a simple surface. Put \(P = r(a, b)\).

(a) The **tangent plane** to \(r\) at \(P\) is the plane through \(P\) and perpendicular to \(r_1(a, b) \times r_2(a, b)\).

(b) The **unit normal** to \(r\) at \(P\) is \(n(a, b) = \frac{r_1 \times r_2}{\|r_1 \times r_2\|}\) (evaluated at \((a, b)\)).
\textbf{Remark}: It follows from the proof of Lemma 4.3 that the tangent plane is an intrinsic invariant: Let \( r : \mathcal{U} \to \mathbb{R}^3 \) be a simple surface, \( f : \mathcal{V} \to \mathcal{U} \) a coordinate transformation with \( f(c,d) = (a,b) \). Put \( P = r(a,b) = (r \circ f)(c,d) \). Then the tangent plane to \( r(u_1, u_2) \) at \( P \) is the same as the tangent plane to \( r \circ f(v_1, v_2) \) at \( P \). Similarly, the unit normal to \( r(u_1, u_2) \) at \( P \) is equal to the unit normal to \( r(v_1, v_2) \) at \( P \) up to sign.

\textbf{Definition}: Let \( r : \mathcal{U} \to \mathbb{R}^3 \) be a simple surface and \( P = r(a,b) \). A vector \( X \in \mathbb{R}^3 \) is a \textit{tangent vector} to \( r \) at \( P \) if \( X \) is the velocity vector at \( P \) of some curve in \( r(\mathcal{U}) \). This means, there exist \( \varepsilon > 0 \) and a curve \( \alpha : (-\varepsilon, \varepsilon) \to r(\mathcal{U}) \) such that \( \alpha(0) = P \) and \( \frac{d\alpha}{dt}(0) = X \).

\textbf{Remark}: Another way of describing a curve \( \alpha \) on the surface \( r(\mathcal{U}) \) is by giving \( u_1 \) and \( u_2 \) as functions of a new variable \( t \). So we get two real-valued functions \( \alpha_1(t) \) and \( \alpha_2(t) \) and we put \( \alpha(t) = r(\alpha_1(t), \alpha_2(t)) \).

\textbf{Lemma 4.4} Let \( r : \mathcal{U} \to \mathbb{R}^3 \) be a simple surface and \( P = r(a,b) \). Then the set of all tangent vectors to \( r \) at \( P \) is a vector space.

\textbf{Proof}: Let \( X \) and \( Y \) be tangent vectors to \( r \) at \( P \) and \( \lambda, \mu \in \mathbb{R} \). We have to show that \( \lambda X + \mu Y \) is also a tangent vector to \( r \) at \( P \). Since \( X \) and \( Y \) are tangent vectors to \( r \) at \( P \), there exist curves \( \alpha(t) \) and \( \beta(t) \) in \( r(\mathcal{U}) \) such that \( \alpha(0) = \beta(0) = P \), \( \frac{d\alpha}{dt}(0) = X \) and \( \frac{d\beta}{dt}(0) = Y \). Hence there exist functions \( \alpha_1(t) \), \( \alpha_2(t) \), \( \beta_1(t) \) and \( \beta_2(t) \) such that \( \alpha(t) = r(\alpha_1(t), \alpha_2(t)) \) and \( \beta(t) = r(\beta_1(t), \beta_2(t)) \). Using the chain rule, we get that

\[
X = \left. \frac{d\alpha}{dt} \right|_{t=0} = \left[ \frac{\partial r}{\partial u_1} \frac{d\alpha_1}{dt} + \frac{\partial r}{\partial u_2} \frac{d\alpha_2}{dt} \right]_{t=0} \quad \text{and} \quad Y = \left. \frac{d\beta}{dt} \right|_{t=0} = \left[ \frac{\partial r}{\partial u_1} \frac{d\beta_1}{dt} + \frac{\partial r}{\partial u_2} \frac{d\beta_2}{dt} \right]_{t=0}
\]

Put

\[
\gamma_1(t) = \lambda(\alpha_1(t) - a) + \mu(\beta_1(t) - a) + a \quad \text{and} \quad \gamma_2(t) = \lambda(\alpha_2(t) - b) + \mu(\beta_2(t) - b) + b
\]

Then \( \gamma_1(0) = a \) and \( \gamma_2(0) = b \). Put \( \gamma(t) = r(\gamma_1(t), \gamma_2(t)) \). Then for small values of \( t \), \( \gamma \) is a curve in \( r(\mathcal{U}) \) and \( \gamma(0) = r(a,b) = P \). Using the chain rule, we get that

\[
\left. \frac{d\gamma}{dt} \right|_{t=0} = \left. \left[ \frac{\partial r}{\partial u_1} \frac{d\gamma_1}{dt} + \frac{\partial r}{\partial u_2} \frac{d\gamma_2}{dt} \right] \right|_{t=0}
\]

\[
= \left. \left[ \frac{\partial r}{\partial u_1} \left( \lambda \frac{d\alpha_1}{dt} + \mu \frac{d\beta_1}{dt} \right) + \frac{\partial r}{\partial u_2} \left( \lambda \frac{d\alpha_2}{dt} + \mu \frac{d\beta_2}{dt} \right) \right] \right|_{t=0}
\]

\[
= \lambda \left[ \left. \frac{\partial r}{\partial u_1} \frac{d\alpha_1}{dt} + \frac{\partial r}{\partial u_2} \frac{d\alpha_2}{dt} \right|_{t=0} \right] + \mu \left[ \left. \frac{\partial r}{\partial u_1} \frac{d\beta_1}{dt} + \frac{\partial r}{\partial u_2} \frac{d\beta_2}{dt} \right|_{t=0} \right]
\]

\[
= \lambda X + \mu Y
\]

Hence \( \lambda X + \mu Y \) is a tangent vector to \( r \) at \( P \). \( \square \)

\textbf{Definitions}: Let \( r : \mathcal{U} \to \mathbb{R}^3 \) be a simple surface and \( P = r(a,b) \).

(a) The \( u_1 \)-\textit{curve} at \( P \) is the curve \( \alpha(u_1) = r(u_1, b) \). The \( u_2 \)-\textit{curve} at \( P \) is the curve \( \beta(u_2) = r(a, u_2) \).

(b) A \textit{parametric curve} on \( r \) is a \( u_1 \)-curve or \( u_2 \)-curve at some point \( P \) on \( r \).

\textbf{Theorem 4.5} Let \( r : \mathcal{U} \to \mathbb{R}^3 \) be a simple surface and \( P = r(a,b) \). The set of all tangent vectors to \( r \) at \( P \) is a vector space of dimension two with basis \( \{ r_1(a,b), r_2(a,b) \} \).
Proof: We first show that $r_1(a, b)$ and $r_2(a, b)$ are tangent vectors to $r$ at $P$. Put $\alpha_1(t) = t + a$, $\alpha_2(t) = b$ and $\alpha(t) = r(\alpha_1(t), \alpha_2(t))$. Then $\alpha(t)$ is a curve in $r(U)$ and $\alpha(0) = r(a, b) = P$. Using the chain rule, we get that

$$\frac{d\alpha}{dt} \bigg|_{t=0} = \left[ \frac{\partial r}{\partial u_1} \frac{d\alpha_1}{dt} + \frac{\partial r}{\partial u_2} \frac{d\alpha_2}{dt} \right]_{t=0} = \frac{\partial r}{\partial u_1}(\alpha_1(0), \alpha_2(0)) \cdot 1 + \frac{\partial r}{\partial u_2}(\alpha_1(0), \alpha_2(0)) \cdot 0 = \frac{\partial r}{\partial u_1}(a, b) = r_1(a, b).$$

So $r_1(a, b)$ is a tangent vector to $r$ at $P$. Similarly, $r_2(a, b)$ is a tangent vector to $r$ at $P$.

By Lemma 4.4, the set of all tangent vectors to $r$ at $P$ is a vector space. Since $r$ is a regular surface, $r_1(a, b) \times r_2(a, b) \neq 0$ and so $r_1(a, b)$ and $r_2(a, b)$ are linearly independent. So it is enough to show that every tangent vector to $r$ at $P$ is a linear combination of $r_1(a, b)$ and $r_2(a, b)$.

Let $X$ be a tangent vector to $r$ at $P$. Then there exists a curve $\gamma(t)$ in $r(U)$ such that $\gamma(0) = P$ and $\frac{d\gamma}{dt} \bigg|_{t=0} = X$.

Hence there exists functions $\gamma_1(t)$ and $\gamma_2(t)$ such that $\gamma(t) = r(\gamma_1(t), \gamma_2(t))$. Put $\lambda = \frac{d\gamma_1}{dt}(0)$ and $\mu = \frac{d\gamma_2}{dt}(0)$. Then $\lambda$ and $\mu$ are constants. Using the chain rule, we get that

$$\frac{d\gamma}{dt} \bigg|_{t=0} = \left[ \frac{\partial r}{\partial u_1} \frac{d\gamma_1}{dt} + \frac{\partial r}{\partial u_2} \frac{d\gamma_2}{dt} \right]_{t=0} = \lambda \frac{\partial r}{\partial u_1}(\gamma_1(0), \gamma_2(0)) + \mu \frac{\partial r}{\partial u_2}(\gamma_1(0), \gamma_2(0)) = \lambda r_1(a, b) + \mu r_2(a, b).$$

Hence $X$ is a linear combination of $r_1(a, b)$ and $r_2(a, b)$. □

Lemma 4.6  Let $\mathbf{r} : U \mapsto \mathbb{R}^3$ be a simple surface, $f : \mathcal{V} \mapsto U$ a coordinate transformation and $f(c, d) = (a, b)$. Put $P = r(a, b)$. If $X = \tilde{\lambda} \frac{\partial r}{\partial v_1} + \tilde{\mu} \frac{\partial r}{\partial v_2}$ then $X = \lambda \frac{\partial r}{\partial v_1} + \mu \frac{\partial r}{\partial v_2}$ where

$$\begin{bmatrix} \lambda \\ \mu \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial v_1} & \frac{\partial f_1}{\partial v_2} \\ \frac{\partial f_2}{\partial v_1} & \frac{\partial f_2}{\partial v_2} \end{bmatrix} \begin{bmatrix} \tilde{\lambda} \\ \tilde{\mu} \end{bmatrix} = J(f) \begin{bmatrix} \tilde{\lambda} \\ \tilde{\mu} \end{bmatrix}.$$

Proof: Note that $r(v_1, v_2) = r(f_1(v_1, v_2), f_2(v_1, v_2))$. Using the chain rule, we get that

$$X = \tilde{\lambda} \frac{\partial r}{\partial v_1} + \tilde{\mu} \frac{\partial r}{\partial v_2} = \tilde{\lambda} \left( \frac{\partial r}{\partial u_1} \frac{\partial f_1}{\partial v_1} + \frac{\partial r}{\partial u_2} \frac{\partial f_2}{\partial v_1} \right) + \tilde{\mu} \left( \frac{\partial r}{\partial u_1} \frac{\partial f_1}{\partial v_2} + \frac{\partial r}{\partial u_2} \frac{\partial f_2}{\partial v_2} \right) = \left( \frac{\partial r}{\partial u_1} \frac{\partial f_1}{\partial v_1} + \frac{\partial r}{\partial u_2} \frac{\partial f_1}{\partial v_2} \right) \frac{\partial r}{\partial u_1} + \left( \frac{\partial r}{\partial u_1} \frac{\partial f_2}{\partial v_1} + \frac{\partial r}{\partial u_2} \frac{\partial f_2}{\partial v_2} \right) \frac{\partial r}{\partial u_2}.$$

This is exactly what we get when we work out the matrix multiplication. □

4.2 Surfaces

Definitions:

(a) Let $M$ be a subset of $\mathbb{R}^3$, $P \in M$ and $\varepsilon > 0$. The $\varepsilon$-neighborhood of $P$ is the set $\{Q \in M \mid d(P, Q) < \varepsilon\}$ where $d$ stands for the distance function.

(b) Let $M$ be a subset of $\mathbb{R}^3$, $P \in M$ and $g : M \mapsto \mathbb{R}^2$ a function. Then $g$ is continuous at $P$ if for every open subset $U$ of $\mathbb{R}^2$ with $g(P) \in U$ there exists an $\varepsilon$-neighborhood $\mathcal{M}$ of $P$ with $g(\mathcal{M}) \subset U$.

(c) Let $\mathbf{r} : U \mapsto \mathbb{R}^3$ be a coordinate patch. Then $\mathbf{r}$ is proper if the inverse function $\mathbf{r}^{-1} : r(U) \mapsto U$ is continuous at each point of $\mathbf{r}(U)$.

(d) Let $M$ be a subset of $\mathbb{R}^3$. Then $M$ is a $C^k$ surface if the following holds:

1. For each point $P \in M$ there is a proper $C^k$ coordinate patch whose image is in $M$ and which contains an $\varepsilon$-neighborhood of $P$ for some $\varepsilon > 0$.
2. Suppose that $\mathbf{r} : U \mapsto \mathbb{R}^3$ and $\mathbf{s} : \mathcal{V} \mapsto \mathbb{R}^3$ are two such coordinate patches. Put $U' = \mathbf{r}(U)$ and $\mathcal{V}' = \mathbf{s}(\mathcal{V})$. Then $s^{-1} \circ r : r^{-1}(U' \cap \mathcal{V}') \mapsto s^{-1}(U' \cap \mathcal{V}')$ is a $C^k$ coordinate transformation.
Theorem 4.7  Let \( f : \mathbb{R}^3 \to \mathbb{R}^3 \) be a differentiable function. Put \( M = \{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = 0\} \). Suppose that 
\[
\begin{pmatrix}
\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z}
\end{pmatrix} \neq 0 
\]
for all \((x, y, z) \in M\). Then \( M \) is a surface.

Proof: This proof uses the Implicit Function Theorem from advanced calculus.

4.3 The First Fundamental Form and Arc Length

Definitions:
(a) Let \( r : \mathcal{U} \to \mathbb{R}^3 \) be a proper coordinate patch. We define
\[
g_{ij}(u_1, u_2) = \langle r_i(u_1, u_2), r_j(u_1, u_2) \rangle \text{ or } g_{ij} = \langle r_i, r_j \rangle \text{ for } i, j = 1, 2
\]
\[
g = \det \left( \begin{array}{cc}
g_{11} & g_{12} \\
g_{21} & g_{22}
\end{array} \right) = \det (g_{ij})
\]
\(g^{kl}\) is the \((k, l)\)-th entry of the inverse of \((g_{ij})\) for \(k, l = 1, 2\).
(b) Let \( M \) be a surface and \( P \) a point on \( M \). The tangent space of \( M \) at \( P \) is the set \( T_PM \) of all tangent vectors to \( M \) at \( P \).
(c) The rule which assigns to any two vectors \( X, Y \in T_PM \) their inner product \( \langle X, Y \rangle \) is called the first fundamental form of the surface \( M \). If \( X = X_1 r_1 + X_2 r_2 \) and \( Y = Y_1 r_1 + Y_2 r_2 \) then we have the following formulas for the first fundamental form:
\[
I(X, Y) = \sum_{i,j=1}^{2} g_{ij} X_i Y_j = \begin{bmatrix} X_1 & X_2 \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\
g_{21} & g_{22}\end{bmatrix} \begin{bmatrix} Y_1 \\
Y_2\end{bmatrix}
\]

Remark: Since \((g_{ij})\) is the matrix representing the inner product restricted to the tangent space, we get that \((g_{ij})\) is a non-singular positive definite matrix. So \(g_{11} > 0, g_{22} > 0, g_{12} = g_{21}\) and \(g > 0\).

Lemma 4.8 Let \( r : \mathcal{U} \to \mathbb{R}^3 \) be a proper coordinate patch. Then the following holds:
(a) \( g = ||r_1 \times r_2||^2 \)
(b) \( g^{11} = \frac{g_{22}}{g}, g^{22} = \frac{g_{11}}{g} \) and \( g^{12} = g^{21} = -\frac{g_{12}}{g} \)
(c) \( \sum_{k=1}^{2} g_{ik} g^{kj} = \delta_{ij} \) for \(i, j = 1, 2\)

Proof: (a) Let \( \theta \) be the angle between \( r_1 \) and \( r_2 \). Then
\[
||r_1 \times r_2||^2 = ||r_1||^2 ||r_2||^2 \sin^2(\theta)
\]
\[
= ||r_1||^2 ||r_2||^2 \left( 1 - \cos^2(\theta) \right)
\]
\[
= ||r_1||^2 ||r_2||^2 \left( 1 - \frac{(r_1, r_2)^2}{||r_1||^2 ||r_2||^2} \right)
\]
\[
= ||r_1||^2 ||r_2||^2 - (r_1, r_2)^2
\]
\[
= g_{11}g_{22} - g_{12}g_{21}
\]
\[
= g
\]
(b) Recall that \[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix}
d & -b \\
-c & a
\end{bmatrix}
\]
if \(ad - bc \neq 0\). Since \(g_{12} = g_{21}\), we have that
\[
\begin{bmatrix}
g_{11} & g_{12} \\
g_{21} & g_{22}
\end{bmatrix}^{-1} = \frac{1}{g} \begin{bmatrix}
g_{22} & -g_{12} \\
-g_{21} & g_{11}
\end{bmatrix}
\]
\(g_{ij}\) is the inverse of \((g_{ij})\), we get that this matrix product is \(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\). \(\square\)

**IMPORTANT NOTATIONS:**

Let \(r : \mathcal{U} \mapsto \mathbb{R}^3\) be a proper coordinate patch and \(f : \mathcal{V} \mapsto \mathcal{U}\) a coordinate transformation. Then we use the following notations:

\[
g_{ij} = (r_i, r_j), \quad (g^{kl}) = (g_{ij})^{-1}\text{ and } g = \det(g_{ij})
\]

\[
s = r \circ f, \quad J = J(f) = \left( \frac{\partial u_i}{\partial v_j} \right)\text{ and } J^{-1} = (J(f))^{-1} = J(f^{-1}) = \left( \frac{\partial v_i}{\partial u_j} \right)
\]

\[
\tilde{g}_{\alpha\beta} = (s_\alpha, s_\beta), \quad (\tilde{g}^{\gamma\delta}) = (\tilde{g}_{\alpha\beta})^{-1}\text{ and } \bar{g} = \det(\tilde{g}_{\alpha\beta})
\]

We want to derive a relation between \((g_{ij})\) and \((\tilde{g}_{\alpha\beta})\).

We have that \(r(u_1, u_2) = s(v_1(u_1, u_2), v_2(u_1, u_2))\). Using the chain rule, we get that \(r_i = \sum_{\alpha=1}^{2} s_\alpha \frac{\partial v_\alpha}{\partial u_i}\) for \(i = 1, 2\). Hence

\[
g_{ij} = \langle r_i, r_j \rangle = \left( \sum_{\alpha=1}^{2} s_\alpha \frac{\partial v_\alpha}{\partial u_i} \right) \left( \sum_{\beta=1}^{2} s_\beta \frac{\partial v_\beta}{\partial u_j} \right) = \sum_{\alpha,\beta=1}^{2} \langle s_\alpha, s_\beta \rangle \frac{\partial v_\alpha}{\partial u_i} \frac{\partial v_\beta}{\partial u_j} = \sum_{\alpha,\beta=1}^{2} \tilde{g}_{\alpha\beta} \frac{\partial v_\alpha}{\partial u_i} \frac{\partial v_\beta}{\partial u_j}
\]

We can rewrite this in matrix form:

\[
\begin{pmatrix}
g_{11} & g_{12} \\
g_{21} & g_{22}
\end{pmatrix} = \left( \frac{\partial v_1}{\partial u_1} \frac{\partial v_2}{\partial u_1} \right) \left( \frac{\partial v_1}{\partial u_2} \frac{\partial v_2}{\partial u_2} \right) \left( \tilde{g}_{11} \tilde{g}_{12} \right) \left( \frac{\partial v_1}{\partial u_2} \frac{\partial v_2}{\partial u_2} \right)
\]

or \((g_{ij}) = J^{-1}(\tilde{g}_{\alpha\beta})J^{-1}\)

Taking determinants, we find that

\[
g = \det(g_{ij}) = \det(J^{-1}) \det(\tilde{g}_{\alpha\beta}) \det(J^{-1}) = \bar{g} \det(J^{-1})^2 = \frac{\bar{g}}{\det(J)^2}
\]

Taking inverses, we find that

\[
(g^{kl}) = (g_{ij})^{-1} = J(\tilde{g}_{\alpha\beta})^{-1}J^t = J(\bar{g}^{\gamma\delta})J^t\text{ or } g^{kl} = \sum_{\gamma,\delta=1}^{2} \bar{g}^{\gamma\delta} \frac{\partial u_k}{\partial v_\gamma} \frac{\partial u_l}{\partial v_\delta}\text{ for } k, l = 1, 2
\]

\[
(g^{kl}) = J(\bar{g}^{\gamma\delta})J^t\text{ or } g^{kl} = \sum_{\gamma,\delta=1}^{2} \bar{g}^{\gamma\delta} \frac{\partial u_k}{\partial v_\gamma} \frac{\partial u_l}{\partial v_\delta}\text{ for } k, l = 1, 2
\]

\[
g = \frac{\bar{g}}{\det(J)^2}
\]
Reversing the role of \((u_1,u_2)\) and \((v_1,v_2)\), we get that

\[
\begin{align*}
\langle \tilde{g}_{\alpha\beta} \rangle &= J^t(g_{ij})J \quad \text{or} \quad \tilde{g}_{\alpha\beta} = \sum_{i,j=1}^{2} g_{ij} \frac{\partial u_i}{\partial v_{\alpha}} \frac{\partial u_j}{\partial v_{\beta}} \quad \text{for } \alpha, \beta = 1, 2 \\
\langle \tilde{g}^{\gamma\delta} \rangle &= J^{-1}(g^{kl})J^{-t} \quad \text{or} \quad \tilde{g}^{\gamma\delta} = \sum_{k,l=1}^{2} g^{kl} \frac{\partial v_k}{\partial u_{\gamma}} \frac{\partial v_l}{\partial u_{\delta}} \quad \text{for } \gamma, \delta = 1, 2 \\
\tilde{g} &= g \det(J)^2
\end{align*}
\]

### 4.4 Normal Curvature, Geodesic Curvature and Gauss’s Formulas

**Definitions**: Let \(r : U \mapsto \mathbb{R}^3\) be a coordinate patch.

(a) If \(P = r(a,b)\) and \(M = r(U)\) then the normal space to \(M\) at \(P\) is the vector space \(N_P M\) spanned by \(n(a,b)\). So \(N_P M = \{\lambda n \mid \lambda \in \mathbb{R}\}\).

(b) Let \(\gamma(s) = r(\gamma_1(s), \gamma_2(s))\) be a unit speed curve in \(r(U)\).

\((1)\) The intrinsic normal of \(\gamma\) is \(S(s) = n(\gamma_1(s), \gamma_2(s)) \times T(s)\) or \(S = n \times T\).

\((2)\) The geodesic curvature of \(\gamma\) is \(\kappa_g(s) = \langle \gamma''(s), S(s) \rangle\).

\((3)\) The normal curvature of \(\gamma\) is \(\kappa_n(s) = \langle \gamma''(s), n(s) \rangle\).

**Remarks**:

(a) We have that \(\mathbb{R}^3 = T_P M \oplus N_P M\). So every \(w \in \mathbb{R}^3\) can be written uniquely as \(w = u + v\) with \(u \in T_P M\) and \(v \in N_P M\). Since \(\{r_1, r_2\}\) is a basis for \(T_P M\) and \(n = \frac{r_1 \times r_2}{\|r_1 \times r_2\|}\) is a basis for \(N_P M\), we have that \(\langle u, v \rangle = 0\) for all \(u \in T_P M\) and all \(v \in N_P M\). Also, every \(w \in \mathbb{R}^3\) can be written uniquely as \(w = ar_1 + br_2 + cn\) with \(a,b,c \in \mathbb{R}\).

(b) Note that \(S \in T_P M\). Since \(T \in T_P M\), we have that \(\langle n, T \rangle = 0\). Hence \(\|S\| = 1\). So \(S\) is a unit vector that is tangent to the surface \(M\) but normal to the curve \(\gamma\).

(c) The geodesic curvature is independent of the coordinate patch we’re using: it only depends on the geometric shape of the curve.

**Lemma 4.9** Let \(r : U \mapsto \mathbb{R}^3\) be a coordinate patch and \(\gamma(s) = r(\gamma_1(s), \gamma_2(s))\) a unit speed curve in \(r(U)\). Then the following holds:

\((a)\) \(\gamma''(s) = \kappa_g(s)S(s) + \kappa_n(s)n(s)\)

\((b)\) \(\kappa^2(s) = \kappa_g^2(s) + \kappa_n^2(s)\)

**Proof**:

(a) We write \(\gamma'(s) = X(s) + \lambda(s)n(s)\) where \(X(s) \in T_P M\) and \(\lambda(s) \in \mathbb{R}\) for all \(s\). By the Frenet-Serret equations, we get that \(\gamma''(s) = T'(s) = \kappa(s)N(s)\). So

\[
0 = \langle \kappa(s)N(s), T(s) \rangle = \langle \gamma''(s), T(s) \rangle = \langle X(s) + \lambda(s)n(s), T(s) \rangle = \langle X(s), T(s) \rangle + \lambda(s) \langle n(s), T(s) \rangle = \langle X(s), T(s) \rangle
\]

since \(\langle n(s), T(s) \rangle = 0\). Since \(X(s) \in T_P M\), we have that \(\langle X(s), n(s) \rangle = 0\). So \(X\) is perpendicular to both \(n(s)\) and \(T(s)\). Hence \(X(s)\) is parallel to \(n(s) \times T(s) = S(s)\). So \(X(s) = \mu(s)S(s)\) where \(\mu(s) \in \mathbb{R}\) for all \(s\). Hence \(\gamma''(s) = \mu(s)S(s) + \lambda(s)n(s)\). Note that \(\langle S(s), n(s) \rangle = 0\). So

\[
\kappa_g(s) = \langle \gamma''(s), S(s) \rangle = \mu(s) \quad \text{and} \quad \kappa_n(s) = \langle \gamma''(s), n(s) \rangle = \lambda(s)
\]
Hence \( \gamma''(s) = \kappa_g(s)S(s) + \kappa_n(s)n(s) \).

(b) Since \( \gamma''(s) = (s)N(s) \), we get that \( \kappa^2(s) = \|\gamma''(s)\|^2 \). By (a), we get that
\[
\kappa^2 = \|\gamma''\|^2 = \langle \kappa_gS + \kappa_n n, \kappa_gS + \kappa_n n \rangle = \kappa_g^2 \langle S, S \rangle + 2 \kappa_g \kappa_n \langle S, n \rangle + \kappa_n^2 \langle n, n \rangle = \kappa_g^2 + \kappa_n^2.
\]

Definitions: Let \( r : \mathcal{U} \rightarrow \mathbb{R}^3 \) be a coordinate patch.

(a) We put \( r_{ij} = \frac{\partial r}{\partial u_i \partial u_j} \) for \( i, j = 1, 2 \). Note that \( r_{ij} = r_{ji} \).

(b) The coefficients of the second fundamental form are the functions \( L_{ij} = \langle r_{ij}, n \rangle \) for \( i, j = 1, 2 \). Note that \( L_{ij} = L_{ji} \).

(c) The second fundamental form of the surface \( M \) is the function \( II : TP\times TP \rightarrow \mathbb{R} : (X, Y) \mapsto \sum_{i,j=1}^{2} L_{ij} X_i Y_j \) where \( X = X_1 r_1 + X_2 r_2 \) and \( Y = Y_1 r_1 + Y_2 r_2 \).

(d) The Christoffel symbols are the functions \( \Gamma^k_{ij} = \sum_{l=1}^{2} \langle r_{ij}, r_l \rangle g^{lk} \) for \( i, j, k = 1, 2 \). Note that \( \Gamma^k_{ij} = \Gamma^k_{ji} \).

Proposition 4.10 Let \( r : \mathcal{U} \rightarrow \mathbb{R}^3 \) be a coordinate patch. Then the following holds:

(a) (Gauss’s formulas) \( r_{ij} = L_{ij} n + \sum_{k=1}^{2} \Gamma^k_{ij} r_k \) for \( i, j = 1, 2 \)

(b) If \( \gamma(s) = r(\gamma_1(s), \gamma_2(s)) \) is a unit speed curve, then
\[
\kappa_n = \sum_{i,j=1}^{2} L_{ij} \gamma_i'' \gamma_j'' \text{ and } \kappa_g S = \sum_{k=1}^{2} \left( \gamma_k'' + \sum_{i,j=1}^{2} \Gamma^k_{ij} \gamma_i'' \gamma_j'' \right) r_k
\]

Proof: (a) Pick \( i, j, k \in \{1, 2\} \). Then there exist \( a, b_1, b_2 \in \mathbb{R} \) such that \( r_{ij} = a n + b_1 r_1 + b_2 r_2 = a n + \sum_{m=1}^{2} b_m r_m \).

Note that \( \langle r_m, n \rangle = 0 \) for \( m = 1, 2 \). Hence \( L_{ij} = \langle r_{ij}, n \rangle = \langle a n, n \rangle = a \). Since \( \langle n, r_l \rangle = 0 \) for \( l = 1, 2 \), we get that \( \langle r_{ij}, r_l \rangle = \sum_{m=1}^{2} b_m \langle r_m, r_l \rangle = \sum_{m=1}^{2} b_m g_{ml} \) for \( l = 1, 2 \). Multiplying this by \( g^{lk} \) and summing over \( l \), we get that\n\[
\Gamma^k_{ij} = \sum_{l=1}^{2} \langle r_{ij}, r_l \rangle g^{lk} = \sum_{m=1}^{2} b_m g_{ml} g^{lk} = \sum_{m=1}^{2} b_m \left( \sum_{l=1}^{2} g_{ml} g^{lk} \right). \text{ Note that } \sum_{l=1}^{2} g_{ml} g^{lk} \text{ is the (m, k)-th entry of the matrix } (g_{ml})(g^{lk}).
\]
Since these are inverse matrices, the (m, k)-th entry is \( \delta_{mk} \). So \( \Gamma^k_{ij} = \sum_{m=1}^{2} b_m \delta_{mk} = b_k \). Hence \( r_{ij} = L_{ij} n + \Gamma^k_{ij} r_1 + \Gamma^k_{ij} r_2 \).

(b) Since \( \gamma(s) = r(\gamma_1(s), \gamma_2(s)) \), we get that \( \gamma' = \frac{d\gamma}{ds} = \frac{\partial r}{\partial u_1} \frac{d\gamma_1}{ds} + \frac{\partial r}{\partial u_2} \frac{d\gamma_2}{ds} = r_1 \gamma_1' + r_2 \gamma_2' = \sum_{i=1}^{2} r_i \gamma_i' \). Note that \( r_i(s) = r_i(\gamma_1(s), \gamma_2(s)) \) for \( i = 1, 2 \). Using the chain rule, we get that \( \frac{dr_i}{ds} = \frac{\partial r_i}{\partial u_1} \frac{d\gamma_1}{ds} + \frac{\partial r_i}{\partial u_2} \frac{d\gamma_2}{ds} = r_1 \gamma_1' + r_2 \gamma_2' = \sum_{j=1}^{2} r_{ij} \gamma_j' \) for \( i = 1, 2 \). Hence\n\[
\gamma'' = \frac{d\gamma'}{ds} = \sum_{i=1}^{2} \left( \frac{dr_i}{ds} \gamma_i' + r_i \frac{d\gamma_i'}{ds} \right) = \sum_{i,j=1}^{2} r_{ij} \gamma_i' \gamma_j'' + \sum_{i=1}^{2} r_i \gamma_i'' = \sum_{i,j=1}^{2} r_{ij} \gamma_i' \gamma_j'' + \sum_{k=1}^{2} r_k \gamma_k''
\]
Using Gauss’s formulas, we get that
\[
\gamma'' = \sum_{i,j=1}^{2} \left( L_{ij} n + \sum_{k=1}^{2} \Gamma^k_{ij} r_k \right) \gamma_i' \gamma_j'' + \sum_{k=1}^{2} r_k \gamma_k'' = \left( \sum_{i,j=1}^{2} L_{ij} \gamma_i' \gamma_j'' \right) n + \sum_{k=1}^{2} \left( \gamma_k'' + \sum_{i,j=1}^{2} \Gamma^k_{ij} \gamma_i' \gamma_j'' \right) r_k
\]
By Lemma 4.9(a), we get that

\[ \kappa_n n + \kappa_g S = \gamma'' = \left( \sum_{i,j=1}^{2} L_{ij} \gamma'_i \gamma'_j \right) n + \sum_{k=1}^{2} \left( \gamma''_k + \sum_{i,j=1}^{2} \Gamma^k_{ij} \gamma'_i \gamma'_j \right) r_k \]

Since \( \{n, r_1, r_2\} \) is a basis for \( \mathbb{R}^3 \) and \( S \) is a linear combination of \( r_1 \) and \( r_2 \) (\( S \) is a tangent vector), we get that

\[ \kappa_n = \sum_{i,j=1}^{2} L_{ij} \gamma'_i \gamma'_j \quad \text{and} \quad \kappa_g S = \sum_{k=1}^{2} \left( \gamma''_k + \sum_{i,j=1}^{2} \Gamma^k_{ij} \gamma'_i \gamma'_j \right) r_k \]

Definitions: Let \( r : \mathcal{U} \to \mathbb{R}^3 \) be a coordinate patch.

(a) A property of \( r \) is intrinsic if it only depends on \( (g_{ij}) \).

(b) Let \( \gamma(s) = r(\gamma_1(s), \gamma_2(s)) \) be a unit speed curve in \( r(\mathcal{U}) \). A property of \( \gamma \) is intrinsic if it only depends on \( (g_{ij}) \) and \( \gamma_1 \) and \( \gamma_2 \).

Proposition 4.11 Let \( r : \mathcal{U} \to \mathbb{R}^3 \) be a coordinate patch. Then

\[ \Gamma^k_{ij} = \frac{1}{2} \sum_{l=1}^{2} \left( \frac{\partial g_{il}}{\partial u_j} - \frac{\partial g_{lj}}{\partial u_i} + \frac{\partial g_{lj}}{\partial u_i} \right) g^{lk} \quad \text{for} \quad i, j, k = 1, 2 \]

So the Christoffel symbols are intrinsic.

Proof: We get that

\[ \frac{\partial g_{ij}}{\partial u_l} = \frac{\partial}{\partial u_l} \langle r_i, r_j \rangle = \left( \frac{\partial r_i}{\partial u_l}, r_j \right) + \left( r_i, \frac{\partial r_j}{\partial u_l} \right) = \langle r_{il}, r_j \rangle + \langle r_i, r_{jl} \rangle \quad \text{for} \quad i, j, l = 1, 2 \]

Permuting the indices around, we get that

\[ \frac{\partial g_{ij}}{\partial u_l} = \langle r_{il}, r_j \rangle + \langle r_{il}, r_{jl} \rangle, \quad \frac{\partial g_{jl}}{\partial u_i} = \langle r_{ij}, r_l \rangle + \langle r_i, r_{jl} \rangle \quad \text{and} \quad \frac{\partial g_{ij}}{\partial u_i} = \langle r_{ji}, r_l \rangle + \langle r_j, r_{li} \rangle \]

Hence we find that \( \Gamma^k_{ij} = \sum_{l=1}^{2} \langle r_{ij}, r_l \rangle g^{lk} = \sum_{l=1}^{2} \frac{1}{2} \left( \frac{\partial g_{il}}{\partial u_j} - \frac{\partial g_{lj}}{\partial u_i} + \frac{\partial g_{lj}}{\partial u_i} \right) g^{lk} \). \( \square \)

4.5 Geodesics

Definition: A geodesic on a surface \( M \) is a regular (unit speed) curve on \( M \) with geodesic curvature equal to zero everywhere (so \( \kappa_g \equiv 0 \)).

Proposition 4.12 Let \( \gamma(t) \) be a regular curve on \( M \). Then \( \gamma \) is a geodesic if and only if \( [n, \gamma', \gamma''] \equiv 0 \).

Proof: Let \( s \) be the arc length on \( \gamma \). Using the chain rule, we get that

\[ \dot{\gamma} = \frac{d\gamma}{dt} = \frac{d\gamma}{ds} \frac{ds}{dt} = \gamma' \dot{s} \quad \text{and} \quad \ddot{\gamma} = \frac{d}{dt} (\gamma' \dot{s}) = \gamma'' \dot{s} + \gamma' \ddot{s} \]

Hence we get that \( \dot{\gamma} \times \ddot{\gamma} = (\gamma' \dot{s}) \times (\gamma'' \dot{s} + \gamma' \ddot{s}) = \ddot{s} \dot{s} \gamma' \times \gamma' = \ddot{s} \dot{s} \gamma' \times \gamma' = \ddot{s} \dot{s} \gamma' \times \gamma' \). So

\[ [n, \gamma, \gamma'] = \langle \gamma \times \gamma', n \rangle = \left( \ddot{s} \dot{s} \gamma' \times \gamma', n \right) = \ddot{s} \dot{s} [n, \gamma', \gamma''] = \ddot{s} \dot{s} [n, T, \gamma''] = \ddot{s} \dot{s} [n, T, \gamma''] = \ddot{s} \dot{s} \langle S, \gamma'' \rangle = \ddot{s} \dot{s} \kappa_g \]

Note that \( \dot{s} = \|\dot{\gamma}\| > 0 \). So \( \kappa_g \equiv 0 \) if and only if \( [n, \gamma', \gamma''] \equiv 0 \). \( \square \)
Proposition 4.13 Let \( r : U \to \mathbb{R}^3 \) be a coordinate patch and \( \gamma(s) = r(\gamma_1(s), \gamma_2(s)) \) a unit speed curve on \( r(U) \). Then \( \gamma \) is a geodesic if and only if

\[
\gamma''_k + \sum_{i,j=1}^{2} \Gamma^k_{ij} \gamma'_i \gamma'_j = 0 \quad \text{for } k = 1, 2
\]

Proof: By Proposition 4.10(b), we have that \( \kappa_g S = \sum_{k=1}^{2} \left( \gamma''_k + \sum_{i,j=1}^{2} \Gamma^k_{ij} \gamma'_i \gamma'_j \right) r_k \). Since \( S \) is a unit vector and \( \{r_1, r_2\} \) are linearly independent, we get that \( \kappa_g \equiv 0 \) if and only if \( \gamma''_k + \sum_{i,j=1}^{2} \Gamma^k_{ij} \gamma'_i \gamma'_j = 0 \) for \( k = 1, 2 \). \( \square \)

Proposition 4.14 Let \( \gamma(s) \) be a unit speed curve on a surface \( M \). Then \( \gamma \) is a geodesic if and only if \( \gamma''(s) \) is normal to the surface for all \( s \) (so \( \gamma'' \) is a multiple of \( n \)).

Proof: By Lemma 4.9(a), we have that \( \gamma''(s) = \kappa_g(s) S(s) + \kappa_n(s) n(s) \). Note that \( S(s) \) and \( n(s) \) are linearly independent. So \( \gamma''(s) \) is a multiple of \( n(s) \) for all \( s \) if and only if \( \kappa_g(s) = 0 \) for all \( s \). \( \square \)

Theorem 4.15 Let \( P \) be a point on a surface \( M \), \( X \) a unit tangent vector to \( M \) at \( P \) (so \( X \in T_PM \) and \( \|X\| = 1 \)) and \( s_0 \in \mathbb{R} \). Then there exists a unique geodesic \( \gamma(s) \) such that \( \gamma(s_0) = P \), \( \gamma'(s_0) = X \) and \( \gamma(s) \) is a unit speed curve.

Proof: The proof uses Picard’s Theorem for solving differential equations. \( \square \)

Theorem 4.16 Let \( M \) be a surface, \( P \) and \( Q \) two points on \( M \) and \( \gamma(s) \) a unit speed curve going through \( P \) and \( Q \). If \( \gamma \) is the shortest curve on \( M \) between \( P \) and \( Q \), then \( \gamma \) is a geodesic.

Proof: Assume that \( P = \gamma(a) \) and \( Q = \gamma(b) \) with \( a < b \). Let \( \kappa_g(s) \) be the geodesic curvature of \( \gamma \). Pick \( s_0 \in (a, b) \). We will show that \( \kappa_g(s_0) = 0 \).

Suppose that \( \kappa_g(s_0) \neq 0 \). Then there exist real numbers \( c \) and \( d \) and a coordinate patch \( r : U \to \mathbb{R}^3 \) such that \( a < c < s_0 < d < b \) and \( \kappa_g(s) \neq 0 \) for all \( s \in [c, d] \) and \( \gamma([c, d]) \subset r(U) \). Then \( \gamma(s) = r(\gamma_1(s), \gamma_2(s)) \) for some functions \( \gamma_i(s) : [c, d] \to \mathbb{R} \) for \( i = 1, 2 \). Let \( \lambda(s) : [c, d] \to \mathbb{R} \) be a \( C^2 \) function such that \( \lambda(c) = \lambda(d) = 0 \), \( \lambda(s_0) \neq 0 \) and \( \lambda(s_0) \kappa_g(s) > 0 \) for all \( s \in (c, d) \) (for example \( \lambda(s) = (s-c)(d-s)\kappa_g(s) \) will work if \( \gamma \) is of class \( C^4 \)). Then there exist functions \( v_i(s) : [c, d] \to \mathbb{R} \) for \( i = 1, 2 \) such that \( \lambda(s) S(s) = \sum_{i=1}^{2} v_i(s) r_i(\gamma_1(s), \gamma_2(s)) \). Since \( \{r_1, r_2\} \) are linearly independent and \( \lambda(c) = \lambda(d) = 0 \), we get that \( v_i(c) = v_i(d) = 0 \) for \( i = 1, 2 \). Put \( f(t, s) = r(\gamma_1(s) + t v_1(s), \gamma_2(s) + t v_2(s)) \) for all \( s \in [c, d] \) and \( |t| \) small enough, say \( t \in (-\varepsilon, \varepsilon) \). Note that \( f(t, c) = r(\gamma_1(c), \gamma_2(c)) = \gamma(c) \). Similarly, \( f(t, d) = \gamma(d) \). So for \( t \in (\varepsilon, d) \), \( f(t, s) \) is a curve on \( M \) joining \( \gamma(c) \) and \( \gamma(d) \). Note that \( f(0, s) = \gamma(s) \). Let \( L(t) \) be the length of the curve \( f(t, s) \) between \( \gamma(c) \) and \( \gamma(d) \). Then \( L(t) = \int_{c}^{d} \left\| \frac{\partial f}{\partial s} \right\| ds = \int_{c}^{d} \left( \frac{\partial f}{\partial s} , \frac{\partial f}{\partial s} \right)^{1/2} ds. \) Hence

\[
L'(t) = \frac{d}{dt} \left( \int_{c}^{d} \left( \frac{\partial f}{\partial s} , \frac{\partial f}{\partial s} \right)^{1/2} ds \right) \frac{\partial}{\partial t} \left( \int_{c}^{d} \left( \frac{\partial f}{\partial s} , \frac{\partial f}{\partial s} \right)^{1/2} ds \right) ds = \left( \frac{\partial f}{\partial s} , \frac{\partial f}{\partial s} \right)^{1/2} ds.
\]

Note that \( \left( \frac{\partial f}{\partial s} , \frac{\partial f}{\partial s} \right)^{1/2} \bigg|_{t=0} = \left( \frac{d\gamma}{ds} , \frac{d\gamma}{ds} \right)^{1/2} = 1 \) since \( \gamma \) is a unit speed curve. So

\[
L'(0) = \int_{c}^{d} \left\| \frac{\partial f}{\partial t} \frac{\partial f}{\partial s} \right\|_{t=0} ds = \int_{c}^{d} \left[ \frac{\partial f}{\partial s} \left( \left\| \frac{\partial f}{\partial t} \frac{\partial f}{\partial s} \right\|_{t=0} - \left\| \frac{\partial f}{\partial s} \frac{\partial^2 f}{\partial s^2} \right\|_{t=0} \right) ds \right.
\]

\[
= \left( \frac{\partial f}{\partial s} \frac{\partial f}{\partial s} \right)_{t=0}^{s=d} - \int_{c}^{d} \left( \frac{\partial f}{\partial t} \frac{\partial^2 f}{\partial s^2} \right)_{t=0}^{s=d} ds
\]

26
We get that \( \frac{\partial f}{\partial t}(0, s) = \sum_{i=1}^{2} r_i(\gamma_1(s), \gamma_2(s))v_i(s) = \lambda(s)S(s) \). But \( \lambda(c) = \lambda(d) = 0 \). So \( \left[ \begin{array}{c} \frac{\partial f}{\partial t} \\ \frac{\partial f}{\partial s} \end{array} \right] \bigg|_{t=0}^{s=d} = 0 \). Also, \\
\[
\frac{\partial^2 f}{\partial s^2} \bigg|_{t=0} = \frac{d^2 f(0, s)}{ds^2} = \frac{d^2 \gamma(s)}{ds^2} = \gamma''(s) = \kappa_g(s)S(s) + \kappa_n(s)n(s) \text{ by Lemma 4.9(a). So} \\
\left[ \begin{array}{c} \frac{\partial f}{\partial t} \\ \frac{\partial f}{\partial s} \end{array} \right] \bigg|_{t=0} = \langle \lambda(s)S(s), \kappa_g(s)S(s) + \kappa_n(s)n(s) \rangle = \lambda(s)\kappa_g(s) > 0 \text{ for all } s \in (c, d) \\
\]

Hence \( L'(0) < 0 \). Since \( \gamma \) is the shortest curve on \( M \) joining \( P \) and \( Q \), \( \gamma \) is also the shortest curve on \( M \) joining \( \gamma(c) \) and \( \gamma(d) \). But \( f(0, s) = \gamma(s) \). So \( L(t) \) has a minimum at \( t = 0 \). Hence \( L'(0) = 0 \), a contradiction. 
So \( \kappa_g(s_0) = 0 \). Since \( s_0 \) was arbitrary, we get that \( \kappa_g(s) = 0 \) for all \( s \in (a, b) \).

### 4.6 Parallel Vector Fields

**Definitions**: Let \( M \) be a surface, \( \gamma(t) : [a, b] \rightarrow M \) a regular curve on \( M \) and \( X(t) : [a, b] \rightarrow \mathbb{R}^3 \) a vector field.

(a) \( X(t) \) is a **vector field along** \( \gamma \) if \( X(t) \in T_{\gamma(t)}M \) for all \( t \in [a, b] \).

(b) \( X(t) \) is parallel along \( \gamma(t) \) if \( X(t) \) is a vector field along \( \gamma \) and \( \frac{dX}{dt} \) is perpendicular to \( M \) for all \( t \in [a, b] \) (so \( \frac{dX}{dt} \) is a multiple of \( n(t) \) for all \( t \in [a, b] \)).

**Proposition 4.17**: Let \( r : \mathcal{U} \rightarrow \mathbb{R}^3 \) be a coordinate patch, \( \gamma(t) = r(\gamma_1(t), \gamma_2(t)) \) a regular curve in \( r(\mathcal{U}) \) and \( X(t) = X_1(t)r_1(t) + X_2(t)r_2(t) \) a vector field along \( \gamma \). Then \( X \) is parallel along \( \gamma \) if and only if

\[
\frac{dX_k}{dt} + \sum_{i,j=1}^{2} \Gamma^k_{ij} X_i \frac{d\gamma_j}{dt} = 0 \quad \text{for } k = 1, 2
\]

**Proof**: We have that

\[
X(t) \text{ is parallel along } \gamma \iff \langle \frac{dX}{dt}, r_l \rangle \equiv 0 \text{ for } l = 1, 2 \\
\iff \frac{d}{dt} \left( \sum_{i=1}^{2} X_i r_i \right), r_l \equiv 0 \text{ for } l = 1, 2 \\
\iff \sum_{i=1}^{2} \left( \frac{dX_i}{dt} r_i + X_i \frac{d r_i}{dt} \right), r_l \equiv 0 \text{ for } l = 1, 2 \\
\iff \sum_{i=1}^{2} \left( \frac{dX_i}{dt} \langle r_i, r_l \rangle + X_i \left( \sum_{j=1}^{2} r_{ij} \frac{d\gamma_j}{dt} \right) \right) \equiv 0 \text{ for } l = 1, 2 \\
\iff \sum_{i=1}^{2} \frac{dX_i}{dt} g_{il} + \sum_{i,j=1}^{2} \langle r_{ij}, r_l \rangle X_i \frac{d\gamma_j}{dt} \equiv 0 \text{ for } l = 1, 2 \quad (*)
\]

Suppose first that \( X(t) \) is parallel along \( \gamma \). Then \( (*) \) holds. Multiplying \( (*) \) by \( g^{lk} \) and summing over \( l \), we get that

\[
0 = \sum_{i,l=1}^{2} \frac{dX_i}{dt} g_{il} g^{lk} + \sum_{i,j,l=1}^{2} \langle r_{ij}, r_l \rangle g^{lk} X_i \frac{d\gamma_j}{dt} = \sum_{i=1}^{2} \frac{dX_i}{dt} Y_{ik} + \sum_{i,j=1}^{2} \Gamma^k_{ij} X_i \frac{d\gamma_j}{dt} = \frac{dX_k}{dt} + \sum_{i,j=1}^{2} \Gamma^k_{ij} X_i \frac{d\gamma_j}{dt} \text{ for } k = 1, 2
\]
Suppose next that \( \frac{dX_k}{dt} + \sum_{i,j=1}^2 \Gamma_{kj}^{i} X_i \frac{d\gamma_j}{dt} = 0 \) for \( k = 1, 2 \). Multiplying this by \( g_{kl} \) and summing over \( k \) we get that

\[
0 = \sum_{k=1}^2 \frac{dX_k}{dt} g_{kl} + \sum_{i,j,k=1}^2 \Gamma_{kj}^{i} g_{kl} X_i \frac{d\gamma_j}{dt} \quad \text{for} \quad l = 1, 2
\]

\[
= \sum_{i=1}^2 \frac{dX_i}{dt} g_{il} + \sum_{i,j,k,p=1}^2 \langle r_{ij}, r_{kp} \rangle g_{lk} g_{kl} X_i \frac{d\gamma_j}{dt} \quad \text{for} \quad l = 1, 2
\]

\[
= \sum_{i=1}^2 \frac{dX_i}{dt} g_{il} + \sum_{i,j,p=1}^2 \langle r_{ij}, r_{i} \rangle \delta_{pl} X_i \frac{d\gamma_j}{dt} \quad \text{for} \quad l = 1, 2
\]

So (*) holds. Hence \( X(t) \) is parallel along \( \gamma \).

**Theorem 4.18** Let \( M \) be a surface, \( \gamma(t) : [a, b] \to \mathbb{R}^3 \) a regular curve on \( M \), \( t_0 \in [a, b] \) and \( \check{X} \in T_{\gamma(t_0)}M \). Then there exists a unique vector field \( X(t) : [a, b] \to \mathbb{R}^3 \) that is parallel along \( \gamma \) with \( X(t_0) = \check{X} \).

**Proof** :

**Definition** : That unique vector field \( X(t) \) is called the parallel translate of \( \check{X} \) along \( \gamma \).

**Proposition 4.19** Let \( M \) be a surface and \( \gamma(t) : [a, b] \to \mathbb{R}^3 \) a regular curve on \( M \). Then the following holds :

(a) If \( X(t) \) is a vector field parallel along \( \gamma \), then \( \|X(t)\| \) is constant.

(b) If \( X(t) \) and \( Y(t) \) are vector fields parallel along \( \gamma \), then the angle between \( X(t) \) and \( Y(t) \) is constant.

**Proof** : Let \( X(t) \) and \( Y(t) \) be vector fields parallel along \( \gamma \). Since \( Y(t) \in T_{\gamma(t)}M \) and \( \frac{dX}{dt} \) is a multiple of \( n(t) \), we have that \( \langle \frac{dX}{dt}, Y(t) \rangle = 0 \) for all \( t \in [a, b] \). Similarly, \( \langle \frac{dY}{dt}, X(t) \rangle = 0 \) for all \( t \in [a, b] \). Hence

\[
0 = \langle \frac{dX}{dt}, Y \rangle + \langle X, \frac{dY}{dt} \rangle = \frac{d}{dt} \langle X, Y \rangle \quad (*)
\]

(a) Substituting \( Y = X \) into (*), we get that \( 0 = \frac{d}{dt} \langle X, X \rangle = \frac{d}{dt} (\|X(t)\|^2) \). Hence \( \|X(t)\| \) is constant.

(b) Let \( \theta(t) \) be the angle between \( X(t) \) and \( Y(t) \) for all \( t \in [a, b] \). From (*), it follows that \( \langle X(t), Y(t) \rangle \) is constant. But \( \langle X(t), Y(t) \rangle = \|X(t)\| \|Y(t)\| \cos(\theta(t)) \) for all \( t \in [a, b] \). From (a), we get that \( \|X(t)\| \) and \( \|Y(t)\| \) are constant. Hence \( \cos(\theta(t)) \) is constant and so also \( \theta(t) \) is constant.

**Proposition 4.20** Let \( M \) be a surface and \( \gamma(s) : [a, b] \to \mathbb{R}^3 \) a unit-speed curve on \( M \). Then \( \gamma'(s) \) is parallel along \( \gamma \) if and only if \( \gamma''(s) \) is a multiple of \( n(s) \) for all \( s \in [a, b] \). By Proposition 4.14, this happens if and only if \( \gamma \) is a geodesic.

**Proof** : We have that \( \gamma'(s) \) is parallel along \( \gamma \) if and only if \( \gamma''(s) \) is a multiple of \( n(s) \) for all \( s \in [a, b] \). By Proposition 4.14, this happens if and only if \( \gamma \) is a geodesic.
4.7 The Weingarten Map

**Definition**: Let \( M \) be a surface, \( P \) a point on \( M \), \( f \) a differentiable function defined on a neighborhood of \( P \) and \( X \in T_P M \). We define the directional derivative of \( f \) in the direction of \( X \) (notation: \( D_X f \)) as follows:

Let \( \alpha(t) \) be a curve on \( M \) with \( \alpha(0) = P \) and \( \frac{d\alpha}{dt}(0) = X \). Then \( D_X f = \frac{d}{dt}(f \circ \alpha)(0) \).

**Remark**: We will prove in the next proposition that \( D_X f \) is well-defined, namely that \( D_X f \) does not depend on the choice of \( \alpha \).

**Proposition 4.21**: Let \( M \) be a surface, \( P \in M \), \( f \) a differentiable function defined on a neighborhood of \( P \), \( r : \mathcal{U} \to \mathbb{R}^3 \) a coordinate patch of \( M \) with \( r(a,b) = P \) and \( X \in T_P M \). If \( X = \sum_{i=1}^{2} X_i r_i \), then \( D_X f = \sum_{i=1}^{2} X_i \frac{\partial}{\partial u_i} (f \circ r) \) \( (a,b) \).

**Proof**: Let \( \alpha(t) \) be a curve on \( M \) with \( \alpha(0) = P \) and \( \frac{d\alpha}{dt}(0) = X \). Put \( \alpha(t) = r(\alpha_1(t), \alpha_2(t)) \). Then \( (f \circ \alpha)(t) = (f \circ r)(\alpha_1(t), \alpha_2(t)) \). Hence

\[
\frac{d}{dt}(f \circ \alpha)(t) = \sum_{i=1}^{2} \frac{\partial}{\partial u_i} (f \circ r) \mid_{(\alpha_1(t), \alpha_2(t))} \frac{d\alpha_i}{dt}(t) \quad (\ast)
\]

Note that \( r(\alpha_1(0), \alpha_2(0)) = \alpha(0) = P = r(a,b) \). So \( \alpha_1(0) = a \) and \( \alpha_2(0) = b \). Since

\[
\sum_{i=1}^{2} X_i r_i(a,b) = X = \frac{d\alpha}{dt}(0) = \frac{d}{dt}(r(\alpha_1(t), \alpha_2(t))) \big|_{t=0} = \sum_{i=1}^{2} r_i(\alpha_1(0), \alpha_2(0)) \frac{d\alpha_i}{dt}(0) = \sum_{i=1}^{2} \frac{d\alpha_i}{dt}(0) r_i(a,b)
\]

we have that \( \frac{d\alpha_i}{dt}(0) = X_i \) for \( i = 1, 2 \). Substituting \( t = 0 \) into \((\ast)\), we get \( \frac{d}{dt}(f \circ \alpha)(0) = \sum_{i=1}^{2} \frac{\partial}{\partial u_i} (f \circ r) \mid_{(a,b)} X_i \).

**Definitions**: Let \( M \) be a surface and \( P \) a point on \( M \).

(a) Let \( f = (f_1, f_2, f_3) \) be a differentiable, vector-valued function defined on a neighborhood of \( P \) and \( X \in T_P M \). Then the directional derivative of \( f \) in the direction of \( X \) is \( D_X f = (D_X f_1, D_X f_2, D_X f_3) \).

(b) The **Weingarten Map** at \( P \) is the function \( L : T_P M \to \mathbb{R}^3 : X \mapsto -D_X n \)

(c) \( L^{ij} = \sum_{l=1}^{2} g^{il} L_{lj} \) for \( i, j = 1, 2 \) (so \( L^{11} \ L^{12} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} = \begin{bmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{bmatrix} \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}^{-1} \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \)).

**Remark**: Since \( n \) only depends (up to a sign) on the surface \( M \) and not on a particular coordinate patch, the same is true for the Weingarten Map.

**Theorem 4.22**: Let \( r : \mathcal{U} \to \mathbb{R}^3 \) be a coordinate patch and \( P = r(a,b) \). Then the following holds:

(a) The Weingarten map \( L \) is a linear transformation from \( T_P M \) to \( T_P M \).

(b) (Weingarten equations) \( L(r_j) = -\frac{\partial n}{\partial u_j} = -n_j = \sum_{i=1}^{2} L^{ij} r_i \) for \( j = 1, 2 \)

(c) If \( X = \sum_{j=1}^{2} X_j r_j \in T_P M \) then \( L(X) = \sum_{i,j=1}^{2} L^{ij} X_j r_i \). So in terms of the basis \( \{r_1, r_2\} \) of \( T_P M \), we have that

\[
L(X) = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}
\]
We will show that \( L(r_1) = -n_1 \). Put \( \alpha_1(t) = a + t \) and \( \alpha_2(t) = b \). Put \( \alpha(t) = r(\alpha_1(t), \alpha_2(t)) = r(a + t, b) \).

Then \( \alpha(t) \) is a curve on \( r(d) \), \( \alpha(0) = r(a, b) = P \) and \( \frac{d\alpha}{dt}(0) = \sum_{i=1}^{2} r_i(a, b) \frac{d\alpha_i}{dt}(0) = r_1(a, b) \). By definition of directional derivative, we get that

\[
L(r_1) = -D_{r_1} n = -\frac{d}{dt}(n \circ \alpha)(t) \bigg|_{t=0} = -\frac{d}{dt}(n(a + t, b)) \bigg|_{t=0} = -\sum_{i=1}^{2} \frac{\partial n_i}{\partial u_i} \bigg|_{(a,b)} \frac{d\alpha_i}{dt}(0) = -n_1(a, b)
\]

Similarly, we get that \( L(r_2) = -n_2 \).

(a) We have that \( L \) is a linear map from \( T_P M \) to \( \mathbb{R}^3 \). We need to show that the image of \( L \) is contained in \( T_P M \). Since \( L \) is linear and \( \{r_1, r_2\} \) is a basis for \( T_P M \), we only need to show that \( L(r_i) \in T_P M \) for \( i = 1, 2 \). Pick \( i \in \{1, 2\} \). Note that \( \langle n, n \rangle = 1 \). Deriving this with respect to \( u_i \), we get that \( 0 = \langle n_i, n \rangle + \langle n, n_i \rangle \). So \( \langle n, n \rangle = 0 \) and \( n_i \) is perpendicular to \( n \). Hence \( n_i \in T_P M \). So \( L(r_i) = -n_i \in T_P M \), which proves (a).

(b) Pick \( j \in \{1, 2\} \). Then \( L(r_j) = \sum_{i=1}^{2} a_i r_i \) for some \( a_1, a_2 \in \mathbb{R} \). For \( k = 1, 2 \), we have that \( \langle n, r_k \rangle = 0 \). Deriving this with respect to \( u_j \), we get that

\[
0 = \langle n_j, r_k \rangle + \langle n, r_{kj} \rangle = \langle -L(r_j), r_k \rangle + L_{kj} = L_{kj} - \left( \sum_{i=1}^{2} a_i r_i, r_k \right) = L_{kj} - \sum_{i=1}^{2} a_i g_{ik} \text{ for } k = 1, 2
\]

So \( L_{kj} = \sum_{i=1}^{2} a_i g_{ik} \) for \( k = 1, 2 \). Multiplying this by \( g^{kl} \) and summing over \( k \), we get that

\[
L^{ij} = \sum_{k=1}^{2} g^{ik} L_{kj} = \sum_{k=1}^{2} g^{ik} L_{kj} = \sum_{i,k=1}^{2} a_i g_{ik} g^{kl} = \sum_{i=1}^{2} a_i \left( \sum_{k=1}^{2} g_{ik} g^{kl} \right) = \sum_{i=1}^{2} a_i \delta_{il} = a_l \text{ for } l = 1, 2
\]

Hence \( L(r_j) = \sum_{i=1}^{2} a_i r_i = \sum_{i=1}^{2} L^{ij} r_i \).

(c) Suppose that \( X = \sum_{j=1}^{2} X_j r_j \). Then \( L(X) = \sum_{j=1}^{2} X_j L(r_j) = \sum_{i,j=1}^{2} L^{ij} X_j r_i \).}

\[
\text{(c) Suppose that } X = \sum_{j=1}^{2} X_j r_j. \text{ Then } L(X) = \sum_{j=1}^{2} X_j L(r_j) = \sum_{i,j=1}^{2} L^{ij} X_j r_i. \quad \square
\]

Remark: When we were working with regular curves, \( \{T, N, B\} \) was an orthonormal basis for \( \mathbb{R}^3 \) at every point on the curve. The Frenet-Serret equations gave us \( \{T', N', B'\} \) in terms of \( \{T, N, B\} \). With surfaces, we still have that \( \{r_1, r_2, n\} \) is a basis for \( \mathbb{R}^3 \) at every point on the surface. Gauss’s formulas (Proposition 4.10(a)) and the Weingarten equations (Theorem 4.22(b)) are the equivalent of the Frenet-Serret equations: for \( j = 1, 2 \), they give us the partial derivatives of \( \{r_1, r_2, n\} \) in terms of \( \{r_1, r_2, n\} \):

\[
\text{for } j = 1, 2 \text{ we have that } \begin{cases} r_{ij} = L_{ij} n + \Gamma^1_{ij} r_1 + \Gamma^2_{ij} r_2 & \text{for } i = 1, 2 \\
_{ij} = -L_{ij} r_1 - L^{2j} r_2 \end{cases}
\]

4.8 Gaussian Curvature and Normal Curvatures

Definition: Let \( M \) be a surface and \( P \in M \). The Gaussian curvature of \( M \) at \( P \) (notation: \( K \)) is the determinant of the Weingarten map \( L \) at \( P \).

Remark: Since the Weingarten map is determined by \( M \) up to a sign and \( \det(L) = \det(-L) \) for any two-by-two matrix \( A \), we get that the Gaussian curvature \( K \) is uniquely determined by the surface \( M \). What is truly remarkable, is that \( K \) is actually intrinsic: it can be calculated using \( (g_{ij}) \).

Theorem 4.23 (Gauss’s Theorema Egregium) The Gaussian curvature \( K \) of a surface is intrinsic.
Proof: \[ \square \]

Remark: To calculate the Gaussian curvature $K$, we don’t need to know the Weingarten map:

\[
K = \det(L^{ij}) = \det((g^{kl})(L_{ij})) = \det((g_{ij})^{-1})(L_{ij}) = \frac{\det\left(\begin{array}{cc} L_{11} & L_{12} \\ L_{21} & L_{22} \end{array}\right)}{\det\left(\begin{array}{cc} g_{11} & g_{12} \\ g_{21} & g_{22} \end{array}\right)}
\]

Another invariant of a linear transformation are its eigenvalues. The eigenvalues of the Weingarten map $L$ at $P$ are determined by $M$ up to a sign. We will see that these values are bounds for the normal curvature at $P$ of any curve on the surface $M$ through the point $P$.

Let $r: \mathcal{U} \mapsto \mathbb{R}^3$ be a coordinate patch and $P = r(a, b)$. If $X \in T_P M$, we write $X = X_1 r_1 + X_2 r_2$. We use the following notations:

\[
1 = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right], \quad G = \left[\begin{array}{cc} g_{11} & g_{12} \\ g_{21} & g_{22} \end{array}\right], \quad L = \left[\begin{array}{cc} L_{11} & L_{12} \\ L_{21} & L_{22} \end{array}\right] \quad \text{and} \quad W = \left[\begin{array}{cc} L_{11} & L_{12} \\ L_{21} & L_{22} \end{array}\right]
\]

Recall the first and second fundamental form:

- $I(X, Y) = \langle X, Y \rangle = \left[\begin{array}{cc} X_1 & X_2 \end{array}\right] G \left[\begin{array}{c} Y_1 \\ Y_2 \end{array}\right]$ for all $X, Y \in T_P M$
- $II(X, Y) = \left[\begin{array}{cc} X_1 & X_2 \end{array}\right] L \left[\begin{array}{c} Y_1 \\ Y_2 \end{array}\right] = \langle L(X), Y \rangle = \langle X, L(Y) \rangle$ for all $X, Y \in T_P M$.

Since the Weingarten map is self-adjoint, the matrix $W$ will have two real eigenvalues, say $\kappa_1$ and $\kappa_2$ (it is possible that $\kappa_1 = \kappa_2$). If $\alpha$ is a curve in $r(\mathcal{U})$ going through $P$, what are the possible values that the normal curvature of $\alpha$ at $P$ can be? There will be a minimum and maximum value that this normal curvature can take on. If $\alpha$ is a unit speed curve, then the normal curvature of $\alpha$ is $II(\alpha', \alpha')$. So we want to answer the following question:

What are the minimum and maximum values of $II(X, X)$ where $X$ is a unit vector in $T_P M$?

We’ll use the method of Lagrange multipliers to answer this question. Put $f(X_1, X_2, \lambda) = II(X, X) - \lambda(\langle X, X \rangle - 1)$.

Then a minimum/maximum can occur when

\[
\begin{align*}
\frac{\partial f}{\partial X_1} &= 0 \\
\frac{\partial f}{\partial X_2} &= 0 \\
\langle X, X \rangle &= 1
\end{align*}
\]

We have that $f(X_1, X_2, \lambda) = II(X, X) - \lambda \langle X, X \rangle + \lambda = \langle L(X), X \rangle - \langle \lambda X, X \rangle + \lambda = \langle L(X) - \lambda X, X \rangle + \lambda$. In matrix form, this becomes

\[
f(X_1, X_2, \lambda) = \lambda + \left[\begin{array}{cc} X_1 & X_2 \end{array}\right] G(W - \lambda 1) \left[\begin{array}{c} X_1 \\ X_2 \end{array}\right]
\]

Note that $G(W - \lambda 1) = GW - \lambda G = GG^{-1}L - \lambda G = L - \lambda G = \left[\begin{array}{cc} L_{11} - \lambda g_{11} & L_{12} - \lambda g_{12} \\ L_{21} - \lambda g_{21} & L_{22} - \lambda g_{22} \end{array}\right]$ is symmetric since $g_{12} = g_{21}$ and $L_{12} = L_{21}$. Put $G(W - \lambda 1) = \left[\begin{array}{cc} a & b \\ b & d \end{array}\right]$. We get that

\[
\frac{\partial f}{\partial X_1} = \left[\begin{array}{cc} 1 & 0 \end{array}\right] \left[\begin{array}{cc} a & b \\ b & d \end{array}\right] \left[\begin{array}{c} X_1 \\ X_2 \end{array}\right] + \left[\begin{array}{c} X_1 \\ X_2 \end{array}\right] \left[\begin{array}{cc} a & b \\ b & d \end{array}\right] = 2 \left[\begin{array}{cc} 1 & 0 \end{array}\right] \left[\begin{array}{cc} a & b \\ b & d \end{array}\right] \left[\begin{array}{c} X_1 \\ X_2 \end{array}\right]
\]

31
Similarly, we find that \( \frac{\partial f}{\partial X_2} = 2 \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ b & d \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \). We want to solve \( \frac{\partial f}{\partial X_1} = \frac{\partial f}{\partial X_2} = 0 \). So suppose that \( \frac{\partial f}{\partial X_1} = \frac{\partial f}{\partial X_2} = 0 \). Then \( \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a & b \\ b & d \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = G(W - \lambda I) \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \). Hence

\[
\begin{bmatrix} Y_1 & Y_2 \end{bmatrix} G(W - \lambda I) \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0 \text{ for all } Y_1, Y_2 \in \mathbb{R}
\]

Note that \( (W - \lambda I) \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = L(X) - \lambda X \). Hence we get that \( \langle L(X) - \lambda X, Y \rangle = 0 \) for all \( Y \in T_P M \). This is only possible when \( L(X) - \lambda X = 0 \). So \( L(X) = \lambda X \). So we have the following result:

If \( \alpha(s) \) is a unit speed curve on the surface going through the point \( P \), then the normal curvature of \( \alpha \) at the point \( P \) will be maximal/minimal (among all such possible curves \( \alpha \)) if \( \alpha'(s) \) is an eigenvector of the Weingarten map \( L \) at the point \( P \).

What are the extreme values of the normal curvature? Let \( \lambda \) be an eigenvalue of the Weingarten map \( L \) and \( X \) a unit eigenvector for \( \lambda \). If \( \alpha(s) \) is a unit speed curve through \( P \) with \( \alpha'(s) = X \), then the normal curvature of \( \alpha \) is given by

\[
II(X, X) = \langle L(X, X) = \langle \lambda X, X \rangle = \lambda \langle X, X \rangle = \lambda
\]

So the minimum and maximum value of the normal curvature are the eigenvalues of the Weingarten map!

**Theorem 4.24** Let \( r : U \mapsto \mathbb{R}^3 \) be a coordinate patch, \( P = r(a, b), \alpha(t) \) a curve in \( r(U) \) with \( \alpha(0) = P \) and \( \kappa_1 \leq \kappa_2 \) the eigenvalues of the Weingarten map \( L \). Then the following holds:

(a) \( \kappa_1(a, b) \leq \kappa_n(0) \leq \kappa_1(a, b) \)

(b) Let \( i \in \{1, 2\} \). Then \( \kappa_i(a, b) = \kappa_n(0) \) if and only if \( \frac{d\alpha}{dt} \) is an eigenvector for \( \kappa_i \)

(c) If \( \kappa_1(a, b) < \kappa_2(a, b) \) and \( X_i \) is an eigenvector for \( \kappa_i(a, b) \) for \( i = 1, 2 \) then \( \langle X_1, X_2 \rangle = 0 \).

**Proof:** We only have to show (c). We get that

\[
\kappa_1 \langle X_1, X_2 \rangle = \langle \kappa_1 X_1, X_2 \rangle = \langle L(X_1), X_2 \rangle = \langle X_1, L(X_2) \rangle = \langle X_1, \kappa_2 X_2 \rangle = \kappa_2 \langle X_1, X_2 \rangle
\]

So \( (\kappa_1 - \kappa_2) \langle L(X_1), X_2 \rangle = \langle X_1, L(X_2) \rangle = \langle X_1, \kappa_2 X_2 \rangle = \kappa_2 \langle X_1, X_2 \rangle = 0 \). Since \( \kappa_1 \neq \kappa_2 \), we get that \( \langle X_1, X_2 \rangle = 0 \).

\( \square \)

So we have the following result:

*At each point of a surface \( M \) there are two orthogonal directions such that the normal curvature takes its maximum value along one direction and its minimum value along the other direction.*

32