1. For each \( n \in \mathbb{N} \), we define \( f_n : [0, +\infty) \to \mathbb{R} : x \to \begin{cases} 1 & \text{if } n \leq x < n + 1 \\ 0 & \text{elsewhere} \end{cases} \).

Prove that \( \int_{[0, +\infty)} \lim f_n < \lim \int_{[0, +\infty)} f_n \).

This shows that we might have strict inequality in Fatou’s Lemma.

2. Let \( g, f, h : D \to \mathbb{R} \) be measurable functions such that \( g \) and \( h \) are integrable over \( D \) and \( g \leq f \leq h \) a.e. on \( D \). Prove that \( f \) is integrable over \( D \).

3. Let \( g, f, f_n : D \to \mathbb{R} \) be measurable functions for all \( n \geq 1 \) such that \( g \) is integrable over \( D \), \( |f_n| \leq g \) a.e. on \( D \) for all \( n \geq 1 \) and \( \langle f_n \rangle_{n \geq 1} \to f \) a.e. on \( D \). Prove that \( \{\int_E |f_n - f|\}_{n \geq 1} \to 0 \).

Hint: use the Lebesgue Convergence Theorem.

4. Let \( g, f_n : D \to \mathbb{R} \) be measurable functions for all \( n \geq 1 \) such that \( g \) is integrable over \( D \) and \( |f_n| \leq g \) a.e. on \( D \) for all \( n \geq 1 \). Prove that
\[
\int_D \lim f_n \leq \lim \int_D f_n \leq \limsup \int_D f_n
\]

5. Let \( f : E \to \mathbb{R} \) be a measurable function and \( E_n \) a measurable subset of \( E \) for all \( n \geq 1 \) such that \( f \) is integrable over \( E_n \) for all \( n \geq 1 \).

(a) Suppose that \( \sum_{n=1}^{+\infty} \int_{E_n} |f| \) converges. Prove that \( f \) is integrable over \( \bigcup_{n=1}^{+\infty} E_n \) and that
\[
\left| \int_{\bigcup_{n \geq 1} E_n} f \right| \leq \sum_{n=1}^{+\infty} \int_{E_n} |f|.
\]

(b) Give an example where \( \sum_{n=1}^{+\infty} \left| \int_{E_n} f \right| \) converges but \( f \) is not integrable over \( \bigcup_{n=1}^{+\infty} E_n \).

Hint for (a): Put \( B_1 = E_1 \) and \( B_n = E_n \setminus (E_1 \cup \cdots \cup E_{n-1}) \) for \( n \geq 2 \). Show that \( \bigcup_{n \geq 1} E_n = \bigcup_{n \geq 1} B_n \) and \( \int_{B_n} f^\pm \leq \int_{E_n} |f| \) for all \( n \geq 1 \).