1. Let $D \subseteq \mathbb{R}$ be a measurable set with $m(D) < \infty$ and $f$ a measurable function defined on $D$ that is finite a.e. on $D$. Prove that for all $\epsilon > 0$, there exists a measurable subset $A$ of $D$ such that $m(A) < \epsilon$ and $f$ is bounded on $D \setminus A$.

2. For $n \geq 1$, define $f_n : [0, +\infty) \to \mathbb{R}: x \to \begin{cases} 1 - \frac{x}{n} & \text{if } 0 \leq x \leq n \\ 0 & \text{if } x > n \end{cases}$

   (a) Put $f(x) = \lim_{n \to +\infty} f_n(x)$ for all $x \geq 0$. Find (with proof) $f$.

   (b) Prove that there exists an $\epsilon > 0$ with the following property:

   For all subsets $A$ of $[0, +\infty)$ with $m^*(A) < \infty$ and for all $n \geq 1$, there exists $x \in [0, +\infty) \setminus A$ with $|f_n(x) - f(x)| \geq \epsilon$.

   (c) Prove that Egoroff’s Theorem fails for this sequence of measurable functions.

   This shows that the condition ‘$m(E)$ is finite’ is needed in Egoroff’s Theorem.

3. For $n \geq 1$, define $f_n : [0, 1] \to \mathbb{R}: x \to \begin{cases} 1 - nx & \text{if } 0 \leq x \leq \frac{1}{n} \\ 0 & \text{if } x > \frac{1}{n} \end{cases}$

   (a) Put $f(x) = 0$ for all $x \in [0, 1]$. Prove that $\{f_n\}_{n \geq 1}$ converges to $f$ a.e. on $[0, 1]$.

   (b) Prove that $\{f_n\}_{n \geq 1}$ does not converge uniformly to $f$ a.e. on $[0, 1]$ (so you have to show that $\{f_n\}_{n \geq 1}$ does not converge uniformly to $f$ on $[0, 1] \setminus A$ for any $A \subseteq [0, 1]$ with $m(A) = 0$).

   This shows that we cannot ‘improve’ Egoroff’s Theorem to uniform convergence a.e.

4. Prove there exists a measurable subset $D \subseteq [0, 1]$ such $\chi_D \circ \psi^{-1}$ is non-measurable.

   This shows that the composition of two measurable functions does not need to be measurable (in fact, measurable after continuous does not need to be measurable).

5. Let $F \subseteq \mathbb{R}$ be closed and $f : F \to \mathbb{R}$ continuous. Prove there exists a continuous function $g : \mathbb{R} \to \mathbb{R}$ with $f = g$ on $F$. 