1. (10 pts) True or False (prove your answer or give a counter example):

(a) (5 pts) A subset $A$ of $\mathbb{R}$ is bounded if and only if $m^*(A)$ is finite.

(b) (5 pts) The set $\left\{ \frac{n}{m} : m, n \in \mathbb{Z} \text{ and } m \geq 0 \right\}$ is countable.

Solution:

(a) FALSE: Consider $\mathbb{Q}$. Since $\mathbb{Q}$ is countable, we know that $m^*(\mathbb{Q}) = 0$. But $\mathbb{Q}$ is not bounded.

(b) TRUE: The given set is clearly a subset of $\mathbb{Q}$. Since $\mathbb{Q}$ is countable and a subset of a countable set is countable, we have that the given set is countable. □

Remark on (a): One direction is true: if $A$ is bounded then $m^*(A)$ is finite. Indeed, since $A$ is bounded, there exists some $M > 0$ such that $|a| \leq M$ for all $a \in A$. So $A \subseteq [-M, M]$. Using monotonicity and the fact that the outer measure of an interval is the length of that interval, we get that $m^*(A) \leq m^*([-M, M]) = 2M$. So $m^*(A)$ is finite.

2. (15 pts) Let $A$ be any subset of $\mathbb{R}$ and $\{E_n\}_{n=1}^{\infty}$ a sequence of pairwise disjoint measurable subsets of $\mathbb{R}$.

(a) (5 pts) Show that $m^*(A \cap (\bigcup_{i=1}^{\infty} E_i)) \leq \sum_{i=1}^{\infty} m^*(A \cap E_i)$

(b) (5 pts) Show that $\sum_{i=1}^{n} m^*(A \cap E_i) \leq m^*(A \cap (\bigcup_{i=1}^{\infty} E_i))$ for all $n \geq 1$.

(c) (5 pts) Use (a) and (b) to deduce that $m^*(A \cap (\bigcup_{i=1}^{\infty} E_i)) = \sum_{i=1}^{\infty} m^*(A \cap E_i)$

Proof:

(a) Note that $A \cap (\bigcup_{i=1}^{\infty} E_i) = \bigcup_{i=1}^{\infty} (A \cap E_i)$ by the Distributive Law. Hence by countable subadditivity, we get that

$m^*(A \cap (\bigcup_{i=1}^{\infty} E_i)) = m^*(\bigcup_{i=1}^{\infty} (A \cap E_i)) \leq \sum_{i=1}^{\infty} m^*(A \cap E_i)$

(b) Let $n \geq 1$. Note that $A \cap (\bigcup_{i=1}^{n} E_i) \subseteq A \cap (\bigcup_{i=1}^{\infty} E_i)$. So by monotonicity and Lemma 2.14, we get that

$\sum_{i=1}^{n} m^*(A \cap E_i) = m^*(A \cap (\bigcup_{i=1}^{n} E_i)) \leq m^*(A \cap (\bigcup_{i=1}^{\infty} E_i))$

(c) From (b), we have that

$\sum_{i=1}^{n} m^*(A \cap E_i) \leq m^*(A \cap (\bigcup_{i=1}^{\infty} E_i))$ for all $n \geq 1$

Taking the limit as $n \to +\infty$ on both sides (note that the right hand side is independent of $n$), we find

$\sum_{i=1}^{\infty} m^*(A \cap E_i) = \lim_{n \to +\infty} \sum_{i=1}^{n} m^*(A \cap E_i) \leq m^*(A \cap (\bigcup_{i=1}^{\infty} E_i))$

Combining this with (a), we get that

$m^*(A \cap (\bigcup_{i=1}^{\infty} E_i)) = \sum_{i=1}^{\infty} m^*(A \cap E_i)$ □
3. (10 pts) We say that a subset of \( \mathbb{R} \) is of type \( F_\sigma \) if it is the countable union of closed sets and is of type \( G_\delta \) if it is the countable intersection of open sets.

Prove that \([1,3)\) is a set of type \( F_\sigma \) and a set of type \( G_\delta \).

**Proof**: First, we show that
\[
[1,3) = \bigcup_{n=1}^{+\infty} \left[1, 3 - \frac{1}{n}\right].
\]
Pick \( x \in \bigcup_{n \geq 1} \left[1, 3 - \frac{1}{n}\right] \). Then there exists \( n \geq 1 \) with \( x \in \left[1, 3 - \frac{1}{n}\right] \subset [1,3) \).

Pick \( x \in [1,3) \). Then \( x < 3 \) and so \( 3 - x > 0 \). Hence \( \frac{1}{3 - x} > 0 \). Let \( m \in \mathbb{N} \) with \( m > \frac{1}{3 - x} \) (this is possible by Archimedes). Then \( 3 - x > \frac{1}{m} \) and so \( x < 3 - \frac{1}{m} \). Hence
\[
x \in \left[1, 3 - \frac{1}{m}\right] \subseteq \bigcup_{n=1}^{+\infty} \left[1, 3 - \frac{1}{n}\right].
\]

Another way: Since \( 3 - x > 0 \) and \( \lim_{n \to +\infty} \frac{1}{n} = 0 \), there exists \( m \in \mathbb{N} \) with \( 0 < \frac{1}{m} < 3 - x \).

Since \( \left[1, 3 - \frac{1}{n}\right] \) is a closed set for all \( n \in \mathbb{N} \), we get that \([1,3)\) is a countable union of closed sets. So \([1,3)\) is of type \( F_\sigma \).

Next, we show that
\[
[1,3) = \bigcap_{n=1}^{+\infty} \left(1 - \frac{1}{n}, 3\right).
\]
Pick \( x \in [1,3) \). So \( 1 \leq x < 3 \). Hence \( 1 - \frac{1}{n} < 1 \leq x < 3 \) for all \( n \in \mathbb{N} \). Then \( x \in \left(1 - \frac{1}{n}, 3\right) \) for all \( n \in \mathbb{N} \).

Pick \( x \in \bigcap_{n=1}^{+\infty} \left(1 - \frac{1}{n}, 3\right) \). Then \( x \in \left(1 - \frac{1}{n}, 3\right) \) for all \( n \geq 1 \). Hence
\[
1 - \frac{1}{n} < x < 3 \quad \text{for all } n \geq 1
\]
Taking the limit as \( n \to +\infty \), we find \( 1 \leq x < 3 \). So \( x \in [1,3) \).

Since \( \left(1 - \frac{1}{n}, 3\right) \) is an open set for all \( n \in \mathbb{N} \), we get that \([1,3)\) is a countable intersection of open sets. So \([1,3)\) is of type \( G_\delta \).

\[\square\]

4. (15 pts) Prove that \( \mathcal{C} := \{A \subseteq \mathbb{R} \mid m^*(A) = 0 \text{ or } m^*(\tilde{A}) = 0\} \) is a \( \sigma \)-algebra.

**Proof**: Pick \( A \in \mathcal{C} \). If \( m^*(\tilde{A}) = 0 \), then \( \tilde{A} \in \mathcal{C} \). If \( m^*(\tilde{A}) \neq 0 \), then \( 0 = m^*(A) = m^*(\tilde{\tilde{A}}) \) and again \( \tilde{A} \in \mathcal{C} \).

Let \( \{A_n \mid n \in \mathbb{N}\} \) be a countable collection of elements in \( \mathcal{C} \). Suppose first that \( m^*(A_n) = 0 \) for all \( n \in \mathbb{N} \). By countable subadditivity, we get that
\[
0 \leq m^*(\bigcup_{n=1}^{+\infty} A_n) \leq \sum_{n=1}^{+\infty} m^*(A_n) = 0
\]
So \( m^*(\bigcup_{n=1}^{+\infty} A_n) = 0 \) and \( \bigcup_{n=1}^{+\infty} A_n \in \mathcal{C} \). Suppose next that \( m^*(A_k) \neq 0 \) for some \( k \in \mathbb{N} \). Then \( m^*(\complement A_k) = 0 \) since \( A_k \in \mathcal{C} \). By De Morgan’s Law, we get that \( \bigcup_{n=1}^{+\infty} A_n = \bigcap_{n=1}^{+\infty} \complement A_n \subseteq \complement A_k \). By monotonicity

\[
0 \leq m^* \left( \bigcup_{n=1}^{+\infty} A_n \right) \leq m^*(\complement A_k) = 0
\]

So \( m^* \left( \bigcup_{n=1}^{+\infty} A_n \right) = 0 \) and again \( \bigcup_{n=1}^{+\infty} A_n \in \mathcal{C} \).

Hence \( \mathcal{C} \) is a \( \sigma \)-algebra. \( \square \)

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5. (10 pts) For \( A \subseteq \mathbb{R} \), we define

\[
m^{**}(A) = \inf \{ m^*(\mathcal{O}) : \mathcal{O} \text{ is open and } A \subseteq \mathcal{O} \}
\]

How is \( m^{**}(A) \) related to \( m^*(A) \)? Prove your ‘best’ answer (e.g. if you believe that \( m^*(A) \leq m^{**}(A) \) then you need to prove this inequality and prove that we do not always have equality).

**Proof**: We will show that \( m^{**}(A) = m^*(A) \) for all \( A \subseteq \mathbb{R} \). Let \( A \subseteq \mathbb{R} \).

Let \( \{I_k\}_{k \geq 1} \) be a ccoi of \( A \). Put \( \mathcal{O} = \bigcup_{k=1}^{+\infty} I_k \). Then \( \mathcal{O} \) is an open set containing \( A \) and

\[
m^*(\mathcal{O}) = m^*(\bigcup_{k=1}^{+\infty} I_k) \leq \sum_{k=1}^{+\infty} m^*(I_k) = \sum_{k=1}^{+\infty} l(I_k)
\]

by countable subadditivity and the fact that the outer measure of an interval is the length of the interval. By definition of \( m^{**}(A) \), we see that \( m^{**}(A) \leq m^*(\mathcal{O}) \). So we proved

\[
m^{**}(A) \leq \sum_{k=1}^{+\infty} l(I_k)
\]

where \( \{I_k\}_{k \geq 1} \) is an arbitrary ccoi of \( A \).

By definition of \( m^*(A) \), we get that \( m^{**}(A) \leq m^*(A) \).

Let \( \mathcal{O} \) be an open set with \( A \subseteq \mathcal{O} \). By monotonicity, we get that \( m^*(A) \leq m^*(\mathcal{O}) \). Since this is true for all open sets \( \mathcal{O} \) containing \( A \), it follows from the definition of \( m^{**}(A) \) that \( m^*(A) \leq m^{**}(A) \).

Hence \( m^{**}(A) = m^*(A) \). \( \square \)

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6. (10 pts) Let \( A, B \subseteq \mathbb{R} \) and \( x \in \mathbb{R} \). Prove the following:

(a) (5 pts) \( A \cap (B + x) = ((A - x) \cap B) + x \)

(b) (5 pts) \( \widetilde{A + x} = \widetilde{A} + x \)

**Proof**: (a) Let \( y \in A \cap (B + x) \). Then \( y \in A \) and so \( y - x \in A - x \). Also, \( y \in B + x \). So \( y = b + x \) for some \( b \in B \). Hence \( y - x = b \in B \). Thus \( y - x \in (A - x) \cap B \). Since \( y = (y - x) + x \), we get that \( y \in ((A - x) \cap B) + x \). So \( A \cap (B + x) \subseteq ((A - x) \cap B) + x \).

Using this result with \( A \leftrightarrow B \) and \( x \leftrightarrow -x \), we get that

\[
B \cap (A - x) \subseteq ((B + x) \cap A) - x
\]

Hence

\[
(B \cap (A - x)) + x \subseteq ((B + x) \cap A) - x + x = (B + x) \cap A
\]

(b) Let \( y \in \mathbb{R} \). Then
\[ y \in \widehat{A} + x \iff y \notin A + x \]
\[ \iff \neg(y \in A + x) \]
\[ \iff \neg(y - x \in A) \]
\[ \iff y - x \notin A \]
\[ \iff y - x \in \widehat{A} \]
\[ \iff y \in \widehat{A} + x \]

7. (10 pts) Let \( f : D \rightarrow \mathbb{R} \) be a function. Prove that \( f \) is continuous over \( D \) if and only if for every closed set \( F \) there exists a closed set \( F^* \) such that \( f^{-1}(F) = D \cap F^* \).

Proof: Suppose first that \( f \) is continuous over \( F \). Let \( F \subseteq \mathbb{R} \) be a closed set. Then \( O := \widehat{F} \) is an open set. By Theorem 1.33, there exists an open set \( O^* \) such that \( f^{-1}(O) = D \cap O^* \). Be careful with the complement notation: we know that if \( g : X \rightarrow Y \) is a function and \( B \subseteq Y \) then \( g^{-1}(\widehat{B}) = g^{-1}(B) \) but \( \widehat{B} = Y \setminus B \) and \( g^{-1}(\widehat{B}) = X \setminus g^{-1}(B) \).

So here we get
\[
D \cap O^* = f^{-1}(O) = D \setminus f^{-1}(F) = D \cap f^{-1}(F)
\]

For this proof, let us reserve the complement notation for the complement in \( \mathbb{R} \). Put \( F^* = \widehat{O^*} \). Then \( F^* \) is closed. Since \( f^{-1}(F) \subseteq D \), we get
\[
f^{-1}(F) = D \setminus f^{-1}(F) = D \cap f^{-1}(F) = D \cap D \cap \widehat{O^*} = D \cap (\widehat{D} \cup \widehat{O^*}) = D \cap (\widehat{D} \cup F^*)
\]

Thus
\[
f^{-1}(F) = D \cap (\widehat{D} \cup F^*) = D \cap (\widehat{D} \cap (D \cap F^*)) = \emptyset \cup (D \cap F^*) = D \cap F^*
\]

Suppose next that for every closed set \( F \), there exists a closed set \( F^* \) such that \( f^{-1}(F) = D \cap F^* \). Let \( O \subseteq \mathbb{R} \) be an open set. Then \( F := \widehat{O} \) is a closed set. Hence there exists a closed set \( F^* \) such that
\[
D \setminus f^{-1}(O) = D \setminus f^{-1}(\widehat{O}) = f^{-1}(F) = D \cap F^*
\]

Put \( O^* = \widehat{F^*} \). Then \( O^* \) is open. Since \( f^{-1}(O) \subseteq D \), we get
\[
f^{-1}(O) = D \setminus f^{-1}(O) = D \cap f^{-1}(O) = D \cap (D \cap F^*) = D \cap D \cap \widehat{F^*} = D \cap (\widehat{D} \cup F^*) = D \cap (\widehat{D} \cup O^*)
\]

Thus
\[
f^{-1}(O) = D \cap (\widehat{D} \cup O^*) = (D \cap \widehat{D}) \cup (D \cap O^*) = \emptyset \cup (D \cap O^*) = D \cap O^*
\]

By Theorem 1.33, \( f \) is continuous over \( D \). \( \square \)