1. Prove the following:

Let \( \{x_n\}_{n \geq 1} \) be a sequence of extended real numbers. Then the following holds:

(a) \( \lim x_n = -\infty \iff \forall M \in \mathbb{R} : \exists n \in \mathbb{N} : \forall k \geq n : x_k \leq M \)

(b) \( \lim x_n = +\infty \iff \forall M \in \mathbb{R} : \forall n \in \mathbb{N} : \exists k \geq n : x_k \geq M \)

(b) \( \lim x_n := \alpha \in \mathbb{R} \iff \begin{cases} \forall \epsilon > 0 : \exists n \in \mathbb{N} : \forall k \geq n : x_k \leq \alpha + \epsilon \\ \forall \epsilon > 0, \forall n \in \mathbb{N} : \exists k \geq n : \alpha - \epsilon \leq x_k \end{cases} \)

\textbf{Proof}:

For \( n \geq 1 \), put \( B_n = \sup_{k \geq n} x_k = \sup \{x_n, x_{n+1}, \ldots\} \). Then \( \lim x_n = \inf_{n \geq 1} B_n \).

(a) Suppose first that \( \lim x_n = -\infty \). Let \( M \in \mathbb{R} \). Since \( \inf_{n \geq 1} B_n = -\infty \), there exists \( n \in \mathbb{N} \) with \( \sup_{k \geq n} x_k = B_n \leq M \). Hence \( x_k \leq B_n \leq M \) for all \( k \geq n \).

Suppose next that the given property hold. Let \( M \in \mathbb{R} \). Then there exists \( n \in \mathbb{N} \) with \( x_k \leq M \) for all \( k \geq n \). Hence \( B_n = \sup_{k \geq n} x_k \leq M \). We proved:

\[ \forall M \in \mathbb{R} : \exists n \in \mathbb{N} : B_n \leq M \]

So \( \lim x_n = \inf_{n \geq 1} B_n = -\infty \).

(b) Suppose first that \( \lim x_n = +\infty \). Then \( \inf_{n \geq 1} B_n = +\infty \). Hence \( B_n = +\infty \) for all \( n \geq 1 \). Let \( M \in \mathbb{R} \) and let \( n \geq 1 \). Since \( B_n = \sup_{k \geq n} x_k = +\infty \), there exists \( k \geq n \) with \( x_k \geq M \).

Suppose next that the given property hold. Let \( M \in \mathbb{R} \) and let \( n \geq 1 \). Then there exists \( k \geq n \) with \( x_k \geq M \). Hence \( B_n = \sup_{k \geq n} x_k \geq M \). Since this is true for all \( n \in \mathbb{N} \), we get that \( \inf_{n \geq 1} B_n \geq M \). Since this is true for all \( M \in \mathbb{R} \), we have that \( \lim x_n = \inf_{n \geq 1} B_n = +\infty \).

(c) Suppose first that \( \lim x_n = \alpha \in \mathbb{R} \). Let \( \epsilon > 0 \). Since \( \inf_{n \geq 1} B_n = \alpha \), there exists \( n \geq 1 \) such that \( \sup_{k \geq n} x_k = B_n < \alpha + \epsilon \). Hence \( x_k < \alpha + \epsilon \) for all \( k \geq n \) and (1) holds. Let \( \epsilon > 0 \) and let \( n \geq 1 \). Then \( \inf_{n \geq 1} B_n = \alpha \leq B_n \). If \( x_k < \alpha - \epsilon \) for all \( k \geq n \) then \( B_n = \sup_{k \geq n} x_k \leq \alpha - \epsilon < \alpha \), a contradiction. Hence there exists \( k \geq n \) with \( \alpha - \epsilon \leq x_k \) and (2) holds.

Suppose next that properties (1) and (2) hold. Let \( n \geq 1 \) and let \( \epsilon > 0 \). By (2), there exists \( k \geq n \) with \( \alpha - \epsilon \leq x_k \). Hence \( \alpha - \epsilon \leq \sup_{k \geq n} x_k = B_n \). Since this is true for all \( \epsilon > 0 \), we have that \( \alpha \leq B_n \). Let \( \epsilon > 0 \). By (1), there exists \( n \geq 1 \) such that \( x_k \leq \alpha + \epsilon \) for all \( k \geq n \). Hence \( B_n = \sup_{k \geq n} x_k \leq \alpha + \epsilon \). We proved:

\[ \begin{align*} 
(i) & \quad \forall n \geq 1 : \alpha \leq B_n \\
(ii) & \quad \forall \epsilon > 0 : \exists n \geq 1 : B_n \leq \alpha + \epsilon
\end{align*} \]

Hence \( \lim x_n = \inf_{n \geq 1} B_n = \alpha \).

\[ \square \]

2. This exercise shows why we consider series with a countable number of terms.

So let \( I \) be an uncountable index set and let \( x_i > 0 \) for all \( i \in I \).

We define

\[ \sum_{i \in I} x_i = \sup \left( \sum_{j \in J} x_j \right) \]

where \( J \subseteq I \) and \( J \) is finite.
Prove that \( \sum_{i \in I} x_i = +\infty. \)

**Proof**: First, we show that \( I = \bigcup_{n=1}^{\infty} \{ i \in I : x_i > \frac{1}{n} \} \). Clearly, \( \bigcup_{n=1}^{\infty} \{ i \in I : x_i > \frac{1}{n} \} \subseteq I. \) So let \( j \in I. \) Then \( x_j > 0 \) and so \( x_j > \frac{1}{m} \) for some \( m \geq 1. \) Hence \( j \in \{ i \in I : x_i > \frac{1}{m} \} \subseteq \bigcup_{n=1}^{\infty} \{ i \in I : x_i > \frac{1}{n} \}. \) Thus \( I \subseteq \bigcup_{n=1}^{\infty} \{ i \in I : x_i > \frac{1}{n} \}. \)

Next, we prove the exercise. If \( \{ i \in I : x_i > \frac{1}{n} \} \) is countable for all \( n \in \mathbb{N}, \) then \( I = \bigcup_{n=1}^{\infty} \{ i \in I : x_i > \frac{1}{n} \} \) is countable by Corollary 1.7, a contradiction. Hence \( \{ i \in I : x_i > \frac{1}{n} \} \) is uncountable (and thus infinite) for some \( n \in \mathbb{N}. \)

Suppose that \( S := \sum_{i \in I} x_i < +\infty. \) Then \( m > nS \) for some \( m \in \mathbb{N}. \) Let \( M \) be a finite subset of \( \{ i \in I : x_i > \frac{1}{n} \} \) with \( m \) elements (note that \( M \) exists since \( \{ i \in I : x_i > \frac{1}{n} \} \) is infinite). Then

\[
S = \sup_{J \subseteq I} \sum_{j \in J} \sum_{i \in M} x_i > m \cdot \frac{1}{n} = \frac{m}{n} > S
\]

a contradiction.

Hence \( \sum_{i \in I} x_i = +\infty. \)

3. Let \( x \in (0, 1) \) and \( p \in \mathbb{N} \) with \( p \geq 2. \) In this exercise, you will prove that there exists a sequence of integers \( \{a_n\}_{n \geq 1} \) such that \( 0 \leq a_n \leq p - 1 \) for all \( n \geq 1 \) and \( x = \sum_{n=1}^{\infty} \frac{a_n}{p^n}. \) So in base \( p, \) \( x = 0.a_1a_2a_3 \ldots. \)

First, you have to come up with a recursive definition of \( a_n. \) To make your definitions look nicer, define \( a_0 = 0. \)

Recall the following definition:

The integral part of a real number \( t \) (notation: \( \lfloor t \rfloor \)) is the largest integer smaller than or equal to \( t.\)

Then we have:

If \( t \in \mathbb{R} \) and \( a \in \mathbb{Z} \) then \( a = \lfloor t \rfloor \iff a \leq t < a + 1. \)

To come up with a formula for \( a_n, \) consider this: if (in decimal form) \( x = 0.7\ldots \) then we have/expect

\[
\frac{7}{10} \leq x < \frac{8}{10}
\]

We can rewrite this as \( 7 \leq 10x < 8. \)

So in general (in decimal form), if \( x = 0.a_1a_2\ldots \) then (it seems) we get that

\[
a_1 \leq 10x < a_1 + 1
\]

and so \( a_1 \leq 10x < a_1 + 1. \) This should allow you to find a ‘formula’ for \( a_1 \) in terms of \( x \) (or \( x \) and \( a_0 \) since we put \( a_0 = 0) \) (and yes, it will involve the integral part function).

Now we can do something similar to define \( a_2 \) in terms of \( x, a_0 \) and \( a_1. \)

(a) Give a recursive definition for \( a_n \) for \( n \geq 1. \)
(b) Prove that \( 0 \leq a_n \leq p - 1 \) and \( 0 \leq x - \frac{a_1}{p} - \frac{a_2}{p^2} - \cdots - \frac{a_n}{p^n} < \frac{1}{p^n} \) for all \( n \in \mathbb{N}. \)
(c) Prove that $x = \sum_{n=1}^{+\infty} \frac{a_n}{p^n}$.

\textbf{Proof} : (a) Put $a_0 = 0$ and define recursively

$$a_n = \left[ p^n \left( x - a_0 - \frac{a_1}{p} - \frac{a_2}{p^2} - \cdots - \frac{a_{n-1}}{p^{n-1}} \right) \right]$$

for $n = 1, 2, 3, \ldots$

(b) We will prove by induction on $n$ that $0 \leq a_n \leq p - 1$ and

$$0 \leq x - a_0 - \frac{a_1}{p} - \frac{a_2}{p^2} - \cdots - \frac{a_n}{p^n} < \frac{1}{p^n}$$

for all $n \geq 0$.

Since $a_0 = 0$ and $x \in (0, 1)$, we have that $0 \leq a_0 \leq p - 1$ and $0 \leq x - a_0 < \frac{1}{p^0}$.

So assume the statement is true for $n = 0, 1, \ldots, k - 1$ for some $k \geq 1$. By induction, we have that

$$0 \leq x - a_0 - \frac{a_1}{p} - \frac{a_2}{p^2} - \cdots - \frac{a_{k-1}}{p^{k-1}} < \frac{1}{p^{k-1}}$$

Hence

$$0 \leq p^k \left( x - a_0 - \frac{a_1}{p} - \frac{a_2}{p^2} - \cdots - \frac{a_{k-1}}{p^{k-1}} \right) < p^k \cdot \frac{1}{p^{k-1}} = p$$

So $0 \leq a_k \leq p - 1$ since

$$a_k = \left[ p^k \left( x - a_0 - \frac{a_1}{p} - \frac{a_2}{p^2} - \cdots - \frac{a_{k-1}}{p^{k-1}} \right) \right]$$

Since $[t] \leq t < [t] + 1$ for all $t \in \mathbb{R}$, we get that

$$a_k \leq p^k \left( x - a_0 - \frac{a_1}{p} - \frac{a_2}{p^2} - \cdots - \frac{a_{k-1}}{p^{k-1}} \right) < a_k + 1$$

So

$$\frac{a_k}{p^k} \leq x - a_0 - \frac{a_1}{p} - \frac{a_2}{p^2} - \cdots - \frac{a_{k-1}}{p^{k-1}} < \frac{a_k + 1}{p^k} = \frac{a_k}{p^k} + \frac{1}{p^k}$$

Hence

$$0 \leq x - a_0 - \frac{a_1}{p} - \frac{a_2}{p^2} - \cdots - \frac{a_{k-1}}{p^{k-1}} - \frac{a_k}{p^k} < \frac{1}{p^k}$$

which finishes the induction proof.

(c) For all $n \geq 0$, put

$$x_n = a_0 + \frac{a_1}{p} + \frac{a_2}{p^2} + \cdots + \frac{a_n}{p^n}$$

Then we proved that

$$0 \leq x - x_n < \frac{1}{p^n}$$

for all $n \geq 0$.

Since $\lim_{n \to +\infty} \frac{1}{p^n} = 0$, it follows from the Pinching Theorem that $\lim_{n \to +\infty} (x - x_n) = 0$. Hence

$$x = \lim_{n \to +\infty} x_n = \lim_{n \to +\infty} \left( a_0 + \frac{a_1}{p} + \frac{a_2}{p^2} + \cdots + \frac{a_n}{p^n} \right) = \lim_{n \to +\infty} \sum_{k=1}^{n} \frac{a_k}{p^k} = \sum_{k=1}^{+\infty} \frac{a_k}{p^k} \quad \Box$$