1. Let \( \{a_n\}_{n \geq 1} \) and \( \{b_n\}_{n \geq 1} \) be sequences of positive real numbers. Prove that 
\[
\lim (a_n b_n) \leq (\lim a_n)(\lim b_n)
\]
provided the product on the righthand side is not of the form \(0 \cdot \infty\).

**Proof:** Since \( a_n \geq 0 \), \( b_n \geq 0 \) and \( a_n b_n \geq 0 \) for all \( n \geq 1 \), we get that \( \lim a_n \geq 0 \), \( \lim b_n \geq 0 \) and \( \lim (a_n b_n) \geq 0 \). If \( \lim a_n = +\infty \) or \( \lim b_n = +\infty \) then \( (\lim a_n)(\lim b_n) = +\infty \) by assumption and the inequality holds.

Hence we may assume that \( \lim a_n = \alpha \in \mathbb{R} \) and \( \lim b_n = \beta \in \mathbb{R} \). Then \( \{a_n b_n\}_{n \geq 1} \) is bounded above. So \( \alpha \) and \( \beta \) are bounded above (this also follows from putting ‘\( \epsilon = 1 \)’ below). Thus \( \gamma := \lim (a_n b_n) \in \mathbb{R} \).

Let \( \epsilon > 0 \). Since \( \lim a_n = \alpha \) and \( \lim b_n = \beta \), we get
\[
\exists n_1 \in \mathbb{N} : \forall k \geq n_1 : a_k < \alpha + \epsilon \quad \exists n_2 \in \mathbb{N} : \forall k \geq n_2 : b_k < \beta + \epsilon
\]

Since \( \lim (a_n b_n) = \gamma \), we have:
\[
\exists k \geq \max\{n_1, n_2\} : \gamma - \epsilon < a_k b_k
\]

For this \( k \), we find
\[
\gamma - \epsilon < a_k b_k < (\alpha + \epsilon)(\beta + \epsilon)
\]

So we proved:
\[
\forall \epsilon > 0 : \gamma - \epsilon < (\alpha + \epsilon)(\beta + \epsilon)
\]

Taking the limit as \( \epsilon \to 0^+ \), we get
\[
\lim (a_n b_n) = \gamma \leq \alpha \beta = (\lim a_n)(\lim b_n)
\]

\( \square \)

**Remark:** We mention another proof. Recall that \( \lim x_n = \lim_{n \to +\infty} \sup_{k \geq n} x_k \). Let \( n \geq 1 \). Then
\[
0 \leq a_n \leq \sup_{k \geq n} a_k \quad \text{and} \quad 0 \leq b_n \leq \sup_{k \geq n} b_k
\]

So
\[
a_n b_n \leq \left( \sup_{k \geq n} a_k \right) \left( \sup_{k \geq n} b_k \right) \quad \text{for all} \quad n \geq 1
\]

Applying the \( \lim \) to both sides, we get
\[
\lim (a_n b_n) \leq \lim \left( \left( \sup_{k \geq n} a_k \right) \left( \sup_{k \geq n} b_k \right) \right)
\]

Since \( \lim_{n \to +\infty} \sup_{k \geq n} a_k = \lim a_n \) and \( \lim_{n \to +\infty} \sup_{k \geq n} b_k = \lim b_n \), we see that the sequence \( \left\{ \left( \sup_{k \geq n} a_k \right) \left( \sup_{k \geq n} b_k \right) \right\}_{n \geq 1} \) converges to \( (\lim a_n)(\lim b_n) \). Hence
\[
\lim \left( \left( \sup_{k \geq n} a_k \right) \left( \sup_{k \geq n} b_k \right) \right) = (\lim a_n)(\lim b_n)
\]
2. Let $I$ be an index set and $O_i$ an open set for all $i \in I$. Prove that there exists a countable subset $J$ of $I$ with 
\[ \bigcup_{i \in J} O_i = \bigcup_{j \in J} O_j. \]

**Proof** : With each \( x \in \bigcup_{i \in I} O_i \), we associate rational numbers \( p_x, q_x \) and an index \( i_x \in I \) such that \( x \in (p_x, q_x) \subseteq O_{i_x} \). So let \( x \in O := \bigcup_{i \in I} O_i \). Then \( x \in O_{i_x} \) for some \( i_x \in I \). Since \( C_{i_x} \) is open, there exists \( r_x > 0 \) with \( (x-r_x, x+r_x) \subseteq O_{i_x} \). Since \( x-r_x < x < x+r_x \), there exist \( p_x, q_x \in Q \) with \( x-r_x < p_x < x < q_x < x+r_x \). Hence \( x \in (p_x, q_x) \subseteq (x-r_x, x+r_x) \subseteq O_{i_x} \).

Note that the collection \( C := \{(p_x, q_x) : x \in O\} \) is countable (the map \( C \to Q \times Q : (p_x, q_x) \mapsto (p_x, q_x) \) is clearly one-to-one and \( Q \times Q \) is countable). For every element \( C \in C \), pick one \( x(C) \in O \) with \( C = (p_{x(C)}, q_{x(C)}) \). We use the notation \( x(C) \) to show that our choice of \( x \) depends on \( C \). Note that there might be uncountably many choices of \( x \) for a specific \( C \); we just pick one. Then the set \( \{i_{x(C)} : C \in C\} \) is a countable subset of \( I \). Clearly, \( \bigcup_{C \in C} O_{i_{x(C)}} \subseteq \bigcup_{i \in I} O_i \). Let \( y \in \bigcup_{i \in I} O_i \). Then 
\[ y \in (q_y, p_y) = (p_{x(D)}, q_{x(D)}) \subseteq O_{i_{x(D)}} \subseteq \bigcup_{C \in C} O_{i_{x(C)}} \]
for some \( D \in C \). Hence \( \bigcup_{i \in I} O_i = \bigcup_{C \in C} O_{i_{x(C)}} \). \( \square \)

3. Let \( x \in (0, 1) \) and \( p \in \mathbb{N} \) with \( p \geq 2 \). Recall that we proved in HW2 #3 that there exists a sequence of integers \( \{a_n\}_{n \geq 1} \) such that \( 0 \leq a_n \leq p-1 \) for all \( n \geq 1 \) and \( x = \sum_{n=1}^{\infty} \frac{a_n}{p^n} \). So in base \( p \), \( x = 0.a_1a_2a_3 \ldots \). This is called an **expansion of \( x \) in base \( p \)**. An expansion is **finite** if there exists \( m \in \mathbb{N} \) with \( a_n = 0 \) for all \( n > m \).

Put \( S = \left\{ \frac{q}{p^n} : q, n \in \mathbb{N} \right\} \cap (0, 1) \).

(a) Suppose \( x = 0.a_1a_2a_3 \ldots = 0.b_1b_2b_3 \ldots \) are expansions of \( x \) in base \( p \). Prove that one of the following holds:
   (i) \( \{a_n\}_{n \geq 1} = \{b_n\}_{n \geq 1} \)
   (ii) There exists \( m \in \mathbb{N} \) such that \( a_n = b_n \) for all \( n < m \), \( b_m \leq p-2 \), \( a_m = b_m + 1 \), \( a_n = 0 \) for all \( n > m \) and \( b_n = p-1 \) for all \( n > m \) (or a similar statement with \( a_n \) and \( b_n \) interchanged).

(b) Suppose \( x \notin S \). Prove that the expansion of \( x \) in base \( p \) is unique. Prove that this unique expansion is not finite.

(c) Suppose \( x \in S \). Prove that \( x \) has exactly two expansions in base \( p \): one finite expansion and one non-finite expansion.

**Proof** : (a) We may assume that \( \{a_n\}_{n \geq 1} \neq \{b_n\}_{n \geq 1} \). Then \( a_n \neq b_n \) for at least one \( n \in \mathbb{N} \). Let \( m \) be minimal with \( a_m \neq b_m \). Then \( m \geq 1 \) and \( a_n = b_n \) for all \( 1 \leq n < m \). Since \( a_m \neq b_m \), we have that either \( a_m > b_m \) or \( a_m < b_m \). WLOG, \( a_m > b_m \). Since 
\[ \sum_{n=1}^{\infty} \frac{a_n}{p^n} = x = \sum_{n=1}^{\infty} \frac{b_n}{p^n} \]
we get that
\[ \sum_{n=m+1}^{\infty} \frac{a_n}{p^n} = \sum_{n=m+1}^{\infty} \frac{b_n}{p^n} \]

Since \( a_n, b_n \in \{0, 1, \ldots, p-1\} \), we get
\[ \frac{a_m}{p^m} = \frac{a_m}{p^m} + \sum_{n=m+1}^{\infty} \frac{0}{p^n} \leq \frac{a_m}{p^m} + \sum_{n=m+1}^{\infty} \frac{a_n}{p^n} = \frac{b_m}{p^m} + \sum_{n=m+1}^{\infty} \frac{b_n}{p^n} \leq \frac{b_m}{p^m} + \sum_{n=m+1}^{\infty} \frac{p-1}{p^n} = \frac{b_m + 1}{p^m} \]
using the formula for the limit of a geometric series. So
\[
\frac{a_m}{p^m} \leq \frac{b_n + 1}{p^m}
\]
and we have equality if and only if \(a_n = 0\) and \(b_n = p - 1\) for all \(n > m\). Since \(a_m > b_m\), we have that \(a_m \geq b_m + 1\). It follows that \(a_m = b_m + 1\) and \(a_n = 0\) and \(b_n = p - 1\) for all \(n > m\). Since \(b_m + 1 = a_m \leq p - 1\), we get that \(b_m \leq p - 2\). So (ii) holds.

(b)(c) Note that it follows from (a) that a number \(x\) has at most one finite representation and also at most one infinite representation (indeed, if \(x\) has two different representations then one most be finite and one must be infinite).

(b) Suppose \(x\) has a finite representation, say \(x = 0.b_1 \ldots b_m\). Then
\[
x = \frac{b_1}{p} + \frac{b_2}{p^2} + \cdots + \frac{b_{m-1}}{p^{m-1}} + \frac{b_m}{p^m} = \frac{b_1 p^{m-1} + b_2 p^{m-2} + \cdots + b_{m-1} p + b_m}{p^m} \in S
\]
a contradiction.
So \(x\) does not have a finite representation. By the remark, we get that \(x\) has a unique infinite representation.

(c) Since \(x \in S\), we have that \(x = \frac{q}{p^m}\) for some \(q, m \in \mathbb{N}\). Note that \(0 < q < p^m\). We can write \(q\) in base \(p\):
\[
q = c_0 + c_1 p + c_2 p^2 + \cdots + c_{m-1} p^{m-1}
\]
where \(c_0, c_1, \ldots, c_{m-1} \in \{0, 1, \ldots, p - 1\}\) (since \(q < p^m\) the highest power of \(p\) that can show up in \(q\) is \(p^{m-1}\)). Then
\[
x = \frac{q}{p^m} = \frac{c_0 + c_1 p + c_2 p^2 + \cdots + c_{m-1} p^{m-1}}{p^m} = \frac{c_0}{p^m} + \frac{c_1}{p^{m-1}} + \frac{c_2}{p^{m-2}} + \cdots + \frac{c_{m-1}}{p} = 0.c_{m-1} \ldots c_2 c_1 c_0
\]
in base \(p\).
So \(x\) has a finite representation in base \(p\), say \(x = 0.a_1 a_2 \ldots a_m\) where \(a_m \neq 0\). Note that
\[
\frac{1}{p^m} = \sum_{n=m+1}^{+\infty} \frac{p-1}{p^n}
\]
We see that \(x\) also has an infinite representation:
\[
x = \frac{a_1}{p} + \frac{a_2}{p^2} + \cdots + \frac{a_{m-1}}{p^{m-1}} + \frac{a_m - 1}{p^m} + \frac{p-1}{p^{m+1}} + \frac{p-1}{p^{m+2}} + \cdots
\]
or \(x = 0.a_1 a_2 \ldots a_{m-1}(a_m - 1)(p-1)(p-1)\ldots\). By the remark, these are the only two representations of \(x\). \(\square\)