1. Let $x, y \in \mathbb{R}$ with $x < y$. Prove that there exists an irrational number $u$ with $x < u < y$.

**Proof**: Suppose there does not exist an irrational number $u$ with $x < u < y$. Then $(x, y) \subseteq \mathbb{Q}$. Hence $(x, y)$ is countable, a contradiction since the interval $(x, y)$ is uncountable.

Hence there exists an irrational number $u$ with $x < u < y$. \qed

2. Is $\mathbb{Q}$ open? Is $\mathbb{Q}$ closed? Prove your answers!

**Solution**: We will show that $\mathbb{Q}$ is neither open nor closed.

Pick $x \in \mathbb{Q}$. Then for any $\delta > 0$, we get that $(x - \delta, x + \delta) \ni \text{irrational number}$; so $(x - \delta, x + \delta) \nsubseteq \mathbb{Q}$. Hence $\mathbb{Q}$ is not open.

Pick $x \in \mathbb{Q}$. Then for any $\delta > 0$, we get that $(x - \delta, x + \delta) \ni \text{rational number}$; so $(x - \delta, x + \delta) \nsubseteq \mathbb{Q}$. Hence $\mathbb{Q}$ is not closed.

3. Recall the following definition of point of closure:

$x$ is a point of closure of $E$ if $\forall \delta > 0 : \exists y \in E : |x - y| < \delta$.

Prove that this definition is equivalent to the following:

$O \cap E \neq \emptyset$ for every open set $O$ containing $x$.

**Proof**: Let $x \in \mathbb{R}$.

Suppose first that $\forall \delta > 0 : \exists y \in E : |x - y| < \delta$. We want to show that $O \cap E \neq \emptyset$ for every open set $O$ containing $x$. So let $O$ be an open set containing $x$. Since $O$ is open, we have that $(x - \delta, x + \delta) \subseteq O$ for some $\delta > 0$. By assumption, we get that $|x - y| < \delta$ for some $y \in E$. But then $y \in (x - \delta, x + \delta) \subseteq O$. So $y \in E \cap O$. Hence $O \cap E \neq \emptyset$.

Suppose next that $O \cap E \neq \emptyset$ for every open set $O$ containing $x$. We need to show that $\forall \delta > 0 : \exists y \in E : |x - y| < \delta$. So let $\delta > 0$. Then $O := (x - \delta, x + \delta)$ is an open set containing $x$. By assumption, $O \cap E \neq \emptyset$, say $y \in O \cap E$. Then $y \in E$ and $|x - y| < \delta$. \qed

4. Recall the Nested Set Theorem (Theorem 1.31) we proved in class:

Let $\{F_n\}_{n \geq 1}$ be a descending sequence of non-empty closed subsets of $\mathbb{R}$ (so $F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$) with $F_1$ bounded. Then $\bigcap_{n=1}^{+\infty} F_n \neq \emptyset$.

Give an example that illustrates that the condition ‘$F_1$ is bounded’ is needed.

**Solution**: For $n \in \mathbb{N}$, put $F_n = [n, +\infty)$. Since $\overline{F_n} = (-\infty, n)$ is open, we see that $F_n$ is closed for all $n \in \mathbb{N}$. Clearly, we have that $F_{n+1} = [n + 1, +\infty) \subseteq [n, +\infty) = F_n$ for all $n \in \mathbb{N}$.

Suppose that $\bigcap_{n=1}^{+\infty} F_n \neq \emptyset$, say $x = \bigcap_{n=1}^{+\infty} F_n$. Then $x \in F_n$ and so $x \geq n$ for all $n \in \mathbb{N}$. By Archimedes, there exists $m \in \mathbb{N}$ with $m > x$. Thus $x \notin F_m$, a contradiction.
Hence \( \bigcap_{n=1}^{+\infty} F_n = \emptyset. \)

5. Let \( A \subseteq \mathbb{R} \) with \( m^*(A) = 0 \). Prove that \( m^*(A \cup B) = m^*(B) \) for all \( B \subseteq \mathbb{R} \).

Proof: Since \( B \subseteq A \cup B \), we get that \( m^*(B) \leq m^*(A \cup B) \) by inclusion. By countable subadditivity, we have that \( m^*(A \cup B) \leq m^*(A) + m^*(B) = m^*(B) \). Hence \( m^*(B) = m^*(A \cup B) \).

6. Let \( A = \mathbb{Q} \cap [0, 1] \) and \( \{I_1, \ldots, I_n\} \) a finite collection of open intervals covering \( A \). Prove that \( \sum_{k=1}^{n} l(I_k) \geq 1 \).

Proof: For \( k = 1, 2, \ldots, n \), we denote by \( \overline{I_k} \) the closed interval with the same endpoints as \( I_k \). Note that \( l(\overline{I_k}) = l(I_k) \) for \( 1 \leq k \leq n \) and \( \bigcup_{k=1}^{n} \overline{I_k} \) is closed.

Pick \( x \in [0, 1] \). Suppose that \( x \not\in \bigcup_{k=1}^{n} \overline{I_k} \). Then \( x \in \bigcup_{k=1}^{n} \overline{I_k} \), which is an open set. Hence there exists \( \epsilon > 0 \) such that \( (x-\epsilon, x+\epsilon) \subseteq \bigcup_{k=1}^{n} \overline{I_k} \). Note that there exists a rational number \( q \) such that \( 0 \leq q \leq 1 \) and \( q \in (x-\epsilon, x+\epsilon) \).

So \( q \in A \subseteq \bigcup_{k=1}^{n} \overline{I_k} \subseteq \bigcup_{k=1}^{n} \overline{I_k} \), a contradiction since \( q \in (x-\epsilon, x+\epsilon) \subseteq \bigcup_{k=1}^{n} \overline{I_k} \). Hence \( x \in \bigcup_{k=1}^{n} \overline{I_k} \).

So \( [0, 1] \subseteq \bigcup_{k=1}^{n} \overline{I_k} \). Then by inclusion, countable subadditivity and the fact that the outer measure of an interval is its length, we get that

\[
1 = m^*([0, 1]) \leq m^*(\bigcup_{k=1}^{n} \overline{I_k}) \leq \sum_{k=1}^{n} m^*(\overline{I_k}) = \sum_{k=1}^{n} l(\overline{I_k}) = \sum_{k=1}^{n} l(I_k) \]

\[\square\]