1. Let $X$ be a set, $\mathcal{C}$ a collection of subsets of $X$ and $\mathcal{A}$ the algebra generated by $\mathcal{C}$. Prove that $\mathcal{C}$ and $\mathcal{A}$ generate the same $\sigma$-algebra.

**Proof:** Let $(\mathcal{C})_\sigma$ be the $\sigma$-algebra generated by $\mathcal{C}$ and $(\mathcal{A})_\sigma$ the $\sigma$-algebra generated by $\mathcal{A}$.

Note that $\mathcal{C} \subseteq \mathcal{A} \subseteq (\mathcal{A})_\sigma$. So $(\mathcal{A})_\sigma$ is a $\sigma$-algebra containing $\mathcal{C}$. Since $(\mathcal{C})_\sigma$ is the smallest $\sigma$-algebra containing $\mathcal{C}$, we get that $(\mathcal{C})_\sigma \subseteq (\mathcal{A})_\sigma$.

Since $(\mathcal{C})_\sigma$ is a $\sigma$-algebra containing $\mathcal{C}$ and every $\sigma$-algebra is an algebra, we have that $(\mathcal{C})_\sigma$ is an algebra containing $\mathcal{C}$. But $\mathcal{A}$ is the smallest algebra containing $\mathcal{C}$. Hence $\mathcal{A} \subseteq (\mathcal{C})_\sigma$. So $(\mathcal{C})_\sigma$ is a $\sigma$-algebra containing $\mathcal{A}$. Since $(\mathcal{A})_\sigma$ is the smallest $\sigma$-algebra containing $\mathcal{A}$, we have that $(\mathcal{A})_\sigma \subseteq (\mathcal{C})_\sigma$.

Hence $(\mathcal{C})_\sigma = (\mathcal{A})_\sigma$. □

2. Show that the condition ‘$m(E_1) \neq +\infty$’ is needed in part (b) of Proposition 2.23 on page 31.

**Solution:** Put $E_k = [k, +\infty)$ for all $k \in \mathbb{N}$. Then $E_k$ is measurable, $m(E_k) = +\infty$ and

$$E_{k+1} = [k + 1, +\infty) \subseteq [k, +\infty) = E_k$$

for all $k \in \mathbb{N}$. So $\{E_k\}_{k \geq 1}$ is a descending sequence of measurable sets and $\lim_{k \to +\infty} m(E_k) = +\infty$.

Suppose that $\cap_{k \geq 1} E_k \neq \emptyset$. Let $x \in \cap_{k \geq 1} E_k$. Then $x \in E_k = [k, +\infty)$ for all $k \geq 1$. By Archimedes’ Axiom, there exists $n \in \mathbb{N}$ with $x < n$. So $x \notin E_n$, a contradiction. Hence $\cap_{k \geq 1} E_k = \emptyset$ and so

$$m(\cap_{k \geq 1} E_k) = m(\emptyset) = 0 \neq +\infty = \lim_{k \to +\infty} m(E_k)$$ □

3. Let $E_1, E_2$ be measurable sets. Prove that

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$$

**Proof:** Note that $E_1 \cap E_2$, $E_1 \cup E_2$ and $E_1 \setminus E_2 = E_1 \setminus (E_1 \cap E_2)$ are all measurable sets.

If $m(E_1 \cap E_2) = +\infty$, then it follows from monotonicity that $m(E_1) = m(E_2) = m(E_1 \cup E_2) = +\infty$ since $E_1 \cap E_2 \subseteq E_1, E_2, E_1 \cup E_2$. But then $m(E_1 \cup E_2) = m(E_1 \cap E_2) = m(E_1) + m(E_2)$.

So we may assume that $m(E_1 \cap E_2)$ is finite. Note that

$$E_1 \cup E_2 = E_2 \cup (E_1 \setminus E_2) = E_2 \cup (E_1 \setminus (E_1 \cap E_2))$$

Using countable additivity and excision, we find

$$m(E_1 \cup E_2) = m(E_2) + m(E_1 \setminus (E_1 \cap E_2)) = m(E_2) + m(E_1) - m(E_1 \cap E_2)$$

Hence $m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$. □

There is another (shorter) proof that does not require us to distinguish several cases. Since

$$E_1 \cap E_2 = E_1 \cap (E_2 \setminus E_1) = E_1 \cup (E_2 \cap E_1)$$

it follows from countable additivity and the measurability of $E_1$ that

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2 \cap E_1) + m(E_2 \cap E_1) = m(E_1) + m(E_2)$$
4. Let $D \subseteq \mathbb{R}$ be a Borel set and $f : D \rightarrow \mathbb{R}$ a function. Prove that the collection

$$S = \{ E \subseteq \mathbb{R} : f^{-1}(E) \text{ is a Borel set} \}$$

is a $\sigma$-algebra.

**Proof:** Note that $S \neq \emptyset$: $f^{-1}(\mathbb{R}) = D$ is a Borel set and so $\mathbb{R} \in S$.

Pick $B \in S$. Then

$$f^{-1}(\overline{B}) = f^{-1}(\bar{B}) = D \setminus f^{-1}(B)$$

which is a difference of Borel sets and hence is a Borel set since the collection of Borel sets is a $\sigma$-algebra. So $\overline{B} \in S$.

Pick $B_n \in S$ for all $n \geq 1$. Then

$$f^{-1}(\bigcup_{n=1}^{\infty} B_n) = \bigcup_{n=1}^{\infty} f^{-1}(B_n)$$

which is a countable union of Borel sets and hence is a Borel set since the collection of Borel sets is a $\sigma$-algebra. So $\bigcup_{n=1}^{\infty} B_n \in S$.

Hence $S$ is a $\sigma$-algebra. \qed

5. Let $D \subseteq \mathbb{R}$ be a Borel set and $f : D \rightarrow \mathbb{R}$ a function continuous on $D$. Prove that $f^{-1}(B)$ is a Borel set for all Borel sets $B \subseteq \mathbb{R}$.

**Proof:** Put $\mathcal{S} = \{ E \subseteq \mathbb{R} : f^{-1}(E) \text{ is a Borel set} \}$. It follows from the previous exercise that $\mathcal{S}$ is a $\sigma$-algebra.

Let $\mathcal{O}$ be an open set. Since $f$ is continuous, it follows from Theorem 1.33 that $f^{-1}(\mathcal{O}) = D \cap \mathcal{O}^\ast$ for some open set $\mathcal{O}^\ast$. Since $D$ and $\mathcal{O}^\ast$ are Borel sets and the collection of Borel sets is a $\sigma$-algebra, we get that $f^{-1}(\mathcal{O})$ is a Borel set. So $\mathcal{O} \in \mathcal{S}$.

Hence $\mathcal{S}$ is a $\sigma$-algebra containing all the open sets. But the collection of Borel sets is the smallest $\sigma$-algebra containing all the open sets. Hence every Borel set belongs to $\mathcal{S}$. Let $B$ be a Borel set. Then $B \in \mathcal{S}$. So $f^{-1}(B)$ is a Borel set. \qed

6. A subset of $\mathbb{R}$ is a $G_\delta$-set if it is the intersection of a countable collection of open sets.

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function. Prove that the set of all points at which $f$ is continuous is a $G_\delta$-set.

**Proof:** Put $\mathcal{S} = \{ a \in \mathbb{R} : f \text{ is continuous at } x = a \}$. For $n \in \mathbb{N}$, put

$$D_n = \left\{ a \in \mathbb{R} : \text{there exists an open set } \mathcal{O} \text{ such that } a \in \mathcal{O} \text{ and } |f(x) - f(y)| < \frac{1}{n} \text{ for all } x, y \in \mathcal{O} \right\}$$

First, we prove that $D_n$ is open for all $n \geq 1$. Indeed, let $n \geq 1$. Pick $a \in D_n$. Then there exists an open set $\mathcal{O}$ such that $a \in \mathcal{O}$ and $|f(x) - f(y)| < \frac{1}{n}$ for all $x, y \in \mathcal{O}$. Since $a \in \mathcal{O}$ and $\mathcal{O}$ is open, we get that $(a - \delta, a + \delta) \subseteq \mathcal{O}$ for some $\delta > 0$. Pick $b \in (a - \delta, a + \delta)$. Then $b \in \mathcal{O}$ and $|f(x) - f(y)| < \frac{1}{n}$ for all $x, y \in \mathcal{O}$. So $b \in D_n$. Hence $(a - \delta, a + \delta) \subseteq D_n$ and $D_n$ is open.

Next, we prove that $S = \cap_{n \geq 1} D_n$.

Pick $a \in S$. Pick $n \in \mathbb{N}$. Since $f$ is continuous at $a$, we get

$$\exists \delta > 0 : \forall x \in \mathbb{R} : |x - a| < \delta \implies |f(x) - f(a)| < \frac{1}{2n}$$
Put $O = (a - \delta, a + \delta)$. Then $O$ is an open set and $a \in O$. Pick $x, y \in O$. Then $|x - a| < \delta$ and $|y - a| < \delta$. Hence

$$|f(x) - f(y)| \leq |f(x) - f(a)| + |f(a) - f(y)| < \frac{1}{2n} + \frac{1}{2} = \frac{1}{n}$$

So $a \in D_n$. Since this is true for all $n \in \mathbb{N}$ and all $a \in S$, we get that $S \subseteq \bigcap_{n \geq 1} D_n$.

Pick $a \in \bigcap_{n \geq 1} D_n$. Pick $\epsilon > 0$. Let $n \in \mathbb{N}$ with $\frac{1}{n} < \epsilon$. Since $a \in D_n$, we get that there exists an open set $O$ such that $a \in O$ and $|f(x) - f(y)| < \frac{1}{n}$ for all $x, y \in O$. Since $a \in O$ and $O$ is open, we have that $(a - \delta, a + \delta) \subseteq O$ for some $\delta > 0$. Pick $x \in \mathbb{R}$ with $|x - a| < \delta$. Then $a, x \in (a - \delta, a + \delta) \subseteq O$. Hence

$$|f(x) - f(a)| < \frac{1}{n} < \epsilon$$

So $f$ is continuous at $a$. Thus $a \in S$. Since this is true for all $a \in \bigcap_{n \geq 1} D_n$, we get that $\bigcap_{n \geq 1} D_n \subseteq S$.

Hence $S = \bigcap_{n \geq 1} D_n$.

We proved that $S$ is the countable intersection of open sets. So $S$ is of type $G_\delta$. \qed