1 \ p. 44: 3(a), 7

3(a) Sketch the region onto which the sector \( r \leq 1, \ 0 \leq \theta \leq \frac{\pi}{4} \) is mapped by the transformation \( w = z^2 \).

**Solution:** Since we are considering a sector of a circle in the \( z \) plane, we consider the exponential representation of \( z \), \( z = re^{i\theta} \). Then,

\[
  w = z^2 \quad \text{becomes} \quad w = r^2e^{i(2\theta)}.
\]
So, if we represent \( w \) in exponential notation as \( w = \rho e^{i\phi} \), we get

\[
  \rho = r^2 \quad \text{and} \quad \phi = 2\theta.
\]
So, the points \( z = r_0e^{i\theta_0} \) on a circle \( r = r_0 \) in the \( z \) plane are transformed into points \( w = r_0^2e^{i(2\theta)} \) on the circle \( \rho = r_0^2 \). Since we consider only the wedge in which \( 0 \leq \theta \leq \frac{\pi}{4} \), this is transformed into the wedge of radius \( \rho = r^2 \), with \( 0 \leq \phi \leq \frac{\pi}{2} \), for \( 0 < r \leq 1 \). The point \( z = 0 \) maps onto \( w = 0 \).

![Diagram](image)

7. Find the image of the semi-infinite strip \( x \geq 0, 0 \leq y \leq \pi \), under the transformation \( w = e^z \), and label corresponding portions of the boundaries.

**Solution:** Recall: We can write \( w = e^z \) as \( w = e^x e^{iy} \). So, if \( w = \rho e^{i\phi} \), then \( \rho = e^x \) and \( \phi = y \).

So, the vertical line \( x = 0 \) will be mapped to

\[
  w = e^{iy}, \ 0 \leq y \leq \pi,
\]
or the arc of a circle of radius 1, \(0 \leq \phi \leq \pi\).

All other vertical lines \(x = c\) will be mapped to \(w = e^{ce^y}, 0 \leq y \leq \pi\). So,

\[ \rho = e^c \text{ and } 0 \leq \phi \leq \pi. \]

The horizontal rays \(y = c_1\) are mapped to rays \(\phi = c_1\) originating from the arc \(w = e^{i\phi}, 0 \leq \phi \leq \pi\).

So, the ray \(y = 0, x \geq 0\) is mapped to the horizontal ray \(v = 0, \pi \leq u\) and the ray \(y = \pi, x \geq 0\) is mapped to the vertical ray \(u = 0, v \geq \pi\).

2 \hspace{1cm} \text{p. 55-56: 3, 5, 10}

3. Let \(n\) be a positive integer and let \(P(z)\) and \(Q(z)\) be polynomials, where \(Q(z_0) \neq 0\). Use Theorem 2 in Section 16, as well as limits appearing in that section, to find

(a) \(\lim_{z \to z_0} \frac{1}{z^n} (z_0 \neq 0)\)

\[ \text{Solution:} \]

\[ \lim_{z \to z_0} \frac{1}{z^n} = \lim_{z \to z_0} \frac{1}{z_0^n} = \frac{1}{z_0^n}. \]

(b) \(\lim_{z \to i} \frac{iz^3 - 1}{z + i}\)

\[ \text{Solution:} \]

\[ \lim_{z \to i} \frac{iz^3 - 1}{z + i} = \frac{\lim_{z \to i} (iz^3 - 1)}{\lim_{z \to i} (z + i)} = \frac{i \lim_{z \to i} z^3 - \lim_{z \to i} 1}{\lim_{z \to i} z + \lim_{z \to i} 1} = \frac{i(i^3) - 1}{i + 1} = 0. \]

(c) \(\lim_{z \to z_0} \frac{P(z)}{Q(z)}\)

\[ \text{Solution:} \]

\[ \lim_{z \to z_0} \frac{P(z)}{Q(z)} = \frac{\lim_{z \to z_0} P(z)}{\lim_{z \to z_0} Q(z)} = \frac{P(z_0)}{Q(z_0)}. \]
5. Show that the limit of the function \( f(z) = \left( \frac{z}{x} \right)^2 \) as \( z \) tends to 0 does not exist. Do this by letting nonzero points \( z = (x, 0) \) and \( z = (x, x) \) approach the origin.

**Solution:**

\[
f(z) = \left( \frac{z}{x} \right)^2 = \left( \frac{x + iy}{x - iy} \right)^2 = \frac{x^2 - y^2 + 2xyi}{x^2 - y^2 - 2xyi}.
\]

Along \( z = (x, 0), x \neq 0 \):

\[
f(z) = \frac{x^2}{x^2} = 1 \implies \lim_{z \to 0} f(z) = 1.
\]

Along \( z = (x, y), x \neq 0 \):

\[
f(z) = \frac{x^2 - x^2 + 2x^2i}{x^2 - x^2 - 2x^2i} = -1 \implies \lim_{z \to 0} f(z) = -1.
\]

Since a limit is unique, we must conclude that the limit of \( f(z) \) as \( z \) goes to 0 does not exist.

Note: It is not sufficient to simply consider points \( z = (x, 0) \) and \( z = (0, y) \), because when \( z = (0, y) \), \( f(z) = 1 \).

10. Use the theorem in Section 17 to show that

(a) \( \lim_{z \to \infty} \frac{4z^2}{(z - 1)^2} = 4 \)

\[
f(z) = \frac{4z^2}{(z - 1)^2} \implies f\left( \frac{1}{z} \right) = \frac{4\left( \frac{1}{z} \right)^2}{\left( \frac{1}{z} - 1 \right)^2} = \frac{4}{z^2 \left( \frac{1}{z^2} - \frac{2}{z} + 1 \right)} = \frac{4}{1 - 2z + z^2}.
\]

Then,

\[
\lim_{z \to \infty} f(z) = \lim_{z \to 0} f\left( \frac{1}{z} \right) = \lim_{z \to 0} \frac{4}{1 - 2z + z^2} = 4. \checkmark
\]

(b) \( \lim_{z \to 1} \frac{1}{(z - 1)^3} = \infty \)

\[
f(z) = \frac{1}{(z - 1)^3} \implies \frac{1}{f(z)} = (z - 1)^3.
\]
Then,

\[ \lim_{z \to 1} \frac{1}{f(z)} = \lim_{z \to 1} (z - 1)^3 = 0. \]

\[ \implies \lim_{z \to 1} f(z) = \infty. \checkmark \]

(c) \( \lim_{z \to \infty} \frac{z^2 + 1}{z - 1} = \infty \)

\[ f(z) = \frac{z^2 + 1}{z - 1} \]

\[ \implies f \left( \frac{1}{z} \right) = \frac{(\frac{1}{z})^2 + 1}{\frac{1}{z} - 1} = \frac{1 + z^2}{z - z^2}. \]

Then,

\[ \lim_{z \to 0} \frac{1}{f \left( \frac{1}{z} \right)} = \lim_{z \to 0} \frac{z - z^2}{1 + z^2} = 0 \]

\[ \implies \lim_{z \to \infty} f(z) = \infty. \checkmark \]

3 p. 62-3: 3, 1, 5, 7

3. Apply definition (3), Section 19, of derivative to give a direct proof that

\[ \frac{dw}{dz} = -\frac{1}{z^2} \text{ when } w = \frac{1}{z} (z \neq 0). \]

\[ \frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} \]

\[ = \lim_{\Delta z \to 0} \frac{\frac{1}{z + \Delta z} - \frac{1}{z}}{\Delta z} \]

\[ = \lim_{\Delta z \to 0} \frac{z - (z + \Delta z)}{z(z + \Delta z)} \]

\[ = \lim_{\Delta z \to 0} \frac{-\Delta z}{\Delta z} \]

\[ = \lim_{\Delta z \to 0} \frac{-1}{z(z + \Delta z)} \]

\[ = -\frac{1}{z^2}. \checkmark \]

1. Use results in Section 20 to find \( f''(z) \) when
(a) \( f(z) = 3z^2 - 2z + 4 \)

\[
f'(z) = 3 \frac{d}{dz} z^2 - 2 \frac{d}{dz} z + \frac{d}{dz} 4 = 3(2z) - 2.
\]

So, \( f'(z) = 6z - 2 \).

(b) \( f(z) = (1 - 4z^2)^3 \)

\[
f'(z) = 3(1 - 4z^2)^2 \cdot \frac{d}{dz} (1 - 4z^2) = 3(1 - 4z^2)^2(-8z).
\]

So, \( f'(z) = -24z(1 - 4z^2)^2 \).

(c) \( f(z) = \frac{z - 1}{2z + 1} \left( z \neq \frac{1}{2} \right) \)

\[
f'(z) = \left[ \frac{d}{dz} \left( z - 1 \right) \right] \left[ 2z + 1 \right] - \left( z - 1 \right) \left[ \frac{d}{dz} \left( 2z + 1 \right) \right] \frac{1}{\left( 2z + 1 \right)^2}
\]

\[
= 1(2z + 1) - (z - 1)2 \frac{1}{\left( 2z + 1 \right)^2}.
\]

So, \( f'(z) = \frac{3}{(2z + 1)^2} \).

(d) \( f(z) = \frac{(1 + z^2)^4}{z^2} (z \neq 0) \)

\[
f'(z) = \left[ \frac{d}{dz} (1 + z^2)^4 \right] z^2 - (1 + z^2)^4 \left[ \frac{d}{dz} z^2 \right] \frac{1}{z^4}
\]

\[
= 4(1 + z^2)^3(2z) - (1 + z^2)^4(2z) \frac{1}{z^4}
\]

\[
= \frac{8z^3(1 + z^2)^3 - 2z(1 + z^2)^4}{z^4}
\]

\[
= \frac{8z^2(1 + z^2)^3 - 2(1 + z^2)^4}{z^3}.
\]

So, \( f'(z) = \frac{2(1 + z^2)^3(3z^2 - 1)}{z^3} \).

5
5. Derive formula (3), Section 20, for the derivative of the sum of two functions.

\[
\frac{d}{dz} [f(z) + g(z)] = \lim_{\Delta z \to 0} \frac{[(f + g)(z + \Delta z) - (f + g)(z)]}{\Delta z}
\]

\[
= \lim_{\Delta z \to 0} \frac{[(f(z + \Delta z) + g(z + \Delta z) - f(z) + g(z)]}{\Delta z}
\]

\[
= \lim_{\Delta z \to 0} \frac{f(z + \Delta z) + g(z + \Delta z) - f(z) - g(z)}{\Delta z}
\]

\[
= \lim_{\Delta z \to 0} \frac{(f(z + \Delta z) - f(z)) + (g(z + \Delta z) - g(z))}{\Delta z}
\]

\[
= \lim_{\Delta z \to 0} \left( \frac{f(z + \Delta z) - f(z)}{\Delta z} + \frac{g(z + \Delta z) - g(z)}{\Delta z} \right)
\]

\[
= \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} + \lim_{\Delta z \to 0} \frac{g(z + \Delta z) - g(z)}{\Delta z}
\]

\[
= f'(z) + g'(z). \Box
\]

7. Prove that the expression (2), Section 20, for the derivative of \( z^n \) remains valid when \( n \) is a negative integer, provided \( z \neq 0 \).

Let \( f(z) = z^n \), where \( n \) is a negative integer.

Let \( m = -n \). Then, \( m \) is a positive integer and

\[
z^n = z^{-m} = \frac{1}{z^m}.
\]

So, \( f(z) = \frac{1}{z^m} \). Then,

\[
\frac{df}{dz} = \frac{d}{dz} \left( \frac{1}{z^m} \right)
\]

\[
= \left[ \frac{d}{dz} (1) \right] z^m - \left[ \frac{d}{dz} (z^m) \right] (1)
\]

\[
= 0(z^m) - mz^{m-1}
\]

\[
= -m \cdot \frac{1}{z^{2m-m+1}}
\]

\[
= -m \frac{z}{z^{m+1}}
\]

\[
= -m z^{-(m+1)}
\]

\[
= -m z^{-m-1}.
\]

Writing this expression in terms of \( n \) gives

\[
\frac{df}{dz} = nz^{n-1}.
\]

So, \( \frac{d}{dz} z^n = nz^{n-1} \) if \( n \) is a negative integer and \( z \neq 0 \).