1 Isolated Singular Points – Section 68 of Brown and Churchill

Definition.

(1) $z_0$ is a **singular point** of a function $f$ if $f$ is not analytic at $z_0$, but is analytic at some point in every neighborhood of $z_0$.

(2) A singular point $z_0$ is an **isolated singular point** if, in addition, there is a deleted neighborhood $0 < |z - z_0| < \epsilon$ of $z_0$ throughout which $f$ is analytic.

**Example.** Determine all singularities of the following functions, and determine if the singularity is isolated.

(1) $f(z) = \frac{\cos z}{z^2}$

$z^2 = 0$ only if $z = 0$, so $z_0 = 0$ is the only singularity. And $f(z)$ is analytic for all $z \neq 0$. Therefore, $z = 0$ is an isolated singularity.

(2) $f(z) = \frac{e^z - 1}{z^2}$

Clearly, $z = 0$ is the only singularity. Again, $f(z)$ is analytic for all $z \neq 0 \implies z = 0$ is an isolated singular point. Note that

$$\frac{e^z - 1}{z^2} = \left(1 + \frac{z^2}{2!} + \cdots \right) - 1 = \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \cdots .$$

(3) $f(z) = \frac{z}{z^2 + 1}$

$z^2 + 1 = 0$ if and only if $z = \pm i$. In this case, then, $f(z)$ has two singularities, both of which are isolated.
(4) \( f(z) = \frac{1 - \cos z}{z} \)

We can see that \( z = 0 \) is the only singular point, and thus, \( z_0 = 0 \) is an isolated singular point.

\[
f(z) = \frac{1 - \cos z}{z} = \frac{1 - \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \right)}{z} = \frac{z}{2!} - \frac{z^3}{4!} + \cdots.
\]

Note that if we define \( f(0) = 0 \), then \( f(z) \) is analytic at \( z = 0 \).

(5) \( f(z) = e^{\frac{z}{2}} \)

\( z = 0 \) is the only singular point of \( f(z) \) and, thus, it is an isolated singular point. Note that

\[
e^{\frac{z}{2}} = 1 + \frac{1}{2} + \frac{1}{2!} \left(\frac{1}{z^2}\right) + \frac{1}{3!} \left(\frac{1}{z^3}\right) + \cdots.
\]

(6) \( f(z) = \frac{1}{\sinh \left(\frac{1}{z}\right)} \)

\[
\sinh \left(\frac{1}{z}\right) = 0 \text{ if } \frac{1}{z} = n\pi i, \ n = \pm 1, \pm 2, \ldots \text{ (Verify.)}
\]

\[
\implies z = -\frac{1}{n\pi}, \ n \in \mathbb{Z}, \text{ or } z = \frac{1}{n\pi} i, \ n = \pm 1, \pm 2, \ldots.
\]

Therefore, the singular points of \( f(z) \) are \( z = 0, z = \pm \frac{1}{\pi} i, z = \pm \frac{1}{2\pi} i, \ldots \).

\( z = \frac{1}{n\pi} i, \ n = \pm 1, \pm 2, \ldots \) are isolated singular points, since there is a deleted neighborhood of each point throughout which \( f \) is analytic.

\( z = 0 \) is not an isolated singularity, because every deleted \( \epsilon \)-neighborhood of \( z = 0 \) contains at least one point of the form \( z = \frac{1}{n\pi} i \).

Why? If we choose \( n \) so that

\[
\frac{1}{n\pi} < \epsilon
\]

\[
\implies n > \frac{1}{\pi\epsilon},
\]

then \( \frac{1}{n\pi} i \) is inside the deleted \( \epsilon \)-neighborhood of \( z = 0 \).
2 Residues – Section 69 of Brown and Churchill

When $z_0$ is an isolated singular point of $f$, there exists $R_2 > 0$ such that $f$ is analytic on the region $0 < |z - z_0| < R_2$.

$\implies f$ has a Laurent series representation about $z_0$:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}, \quad 0 < |z - z_0| < R_2.$$  

coefficients $b_n$ are defined by

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{-n+1}} \, dz, \quad n = 1, 2, \ldots,$$

where $C$ is any positively oriented simple closed contour around $z_0$ lying in $0 < |z - z_0| < R_2$.

$\implies$ When $n = 1$, we obtain

$$\int_C f(z) \, dz = 2\pi i b_1. \quad (1)$$

The complex number $b_1$, which is the coefficient of $\frac{1}{z - z_0}$ in the Laurent expansion, is called the residue of $f$ at the isolated singular point $z_0$, denoted

$$b_1 = \text{Res}_{z=z_0} f(z).$$

Equation (1) can then be written

$$\int_C f(z) \, dz = 2\pi i \, \text{Res}_{z=z_0} f(z).$$

Examples.

(1) Find $\int_C \frac{1}{z(z-1)} \, dz$, where $C$ is the rectangle with vertices at $\frac{1}{2} \pm i$ and $2 \pm i$.

$f(z) = \frac{1}{z(z-1)}$ has two singularities, $z_0 = 0$ and $z_0 = 1$. 


$z_0 = 1$ is the only singularity inside $C$, so we need to find the Laurent expansion of $f(z)$ about $z_0 = 1$.

$$\frac{1}{z(z - 1)} = \frac{1}{(1 + (z - 1))(z - 1)}$$

$$= \frac{1}{z - 1} \cdot \frac{1}{1 - (z - 1)}$$

$$= \frac{1}{z - 1} \left(1 - (z - 1) + (z - 1)^2 + \cdots \right)$$

$$= \frac{1}{z - 1} - 1 + (z - 1) - \cdots .$$

\[ \implies \text{Res}_{z=1} f(z) = 1 \]

\[ \implies \int_C \frac{1}{z(z - 1)} \, dz = 2\pi i (1) = \left[ 2\pi i \right] . \]

(2) Find $\int_C \frac{1}{z(z - 1)} \, dz$, where $C$ is the circle $|z| = \frac{1}{2}$.

In this case, $z_0 = 0$ is the only singularity inside $C$, so we need the Laurent expansion of $f(z)$ about $z_0 = 0$.

$$\frac{1}{z(z - 1)} = \frac{-1}{z(1-z)}$$

$$= \frac{-1}{z} \left(1 + z + z^2 + \cdots \right)$$

$$= \frac{-1}{z} - 1 - z - \cdots .$$

\[ \implies \text{Res}_{z=0} f(z) = -1 \]

\[ \implies \int_C \frac{1}{z(z - 1)} \, dz = 2\pi i (-1) = \left[ -2\pi i \right] . \]

**NOTE:** Examples (1) and (2) may alternately be done by noting that

$$\frac{1}{z(z - 1)} = \frac{-1}{z} + \frac{1}{z - 1}$$

and using the Cauchy-Goursat Theorem and Cauchy’s Integral Formula to compute the integrals.

(3) Find $\int_C \frac{z}{z^2 + 4z + 4} \, dz$, where $C$ is the circle $|z + 2| = 1$.

- **Singularities:** $z^2 + 4z + 4 = 0 \implies (z + 2)^2 = 0 \implies z = -2$. $z_0 = -2$ is the only singularity, and $z_0 = -2$ is inside $C$. 

• Find the Laurent series expansion of \( f(z) = \frac{z}{z^2 + 4z + 4} \) about \( z_0 = -2 \).

\[
\frac{z}{z^2 + 4z + 4} = \frac{z}{(z + 2)^2} = \frac{z + 2 - 2}{(z + 2)^2} = \frac{1}{z + 2} - \frac{2}{(z + 2)^2}
\]

\( \implies \text{Res}_{z=-2} f(z) = 1 \)

\( \implies \int_C \frac{z}{z^2 + 4z + 4} \, dz = 2\pi i (1) = 2\pi i \).

3 Cauchy’s Residue Theorem – Section 70 of Brown and Churchill

If a function \( f \) is analytic inside a simple closed contour at all but a finite number of points, those singular points must be isolated.

**Theorem 1 (Cauchy’s Residue Theorem).** Let \( C \) be a positively oriented simple closed contour. If \( f \) is analytic inside \( C \) except for a finite number of singular points \( z_k, k = 1, 2, \ldots, n \) inside \( C \), then

\[
\int_C f(z) \, dz = 2\pi i \sum_{k=1}^{n} \text{Res}_{z=z_k} f(z).
\]

The proof is a fairly straightforward application of the Cauchy Integral and Cauchy-Goursat theorems.

**Examples.**

(1) Evaluate \( \int_C \frac{1}{z(z + 1)} \, dz \), where \( C \) is the circle \(|z| = 2|\).

The singularities of \( f(z) = \frac{1}{z(z + 1)} \) are \( z_0 = 0 \) and \( z_0 = 1 \), both of which are inside \( C \) \( \implies \)
we need to determine \( \text{Res}_{z=0} f(z) \) and \( \text{Res}_{z=-1} f(z) \).
\( z_0 = 0:\)

\[
\frac{1}{z(z+1)} = \frac{1}{z} \cdot \frac{1}{1-(-z)} \\
= \frac{1}{z} \left(1 - z + z^2 - z^3 + \cdots\right) \\
= \frac{1}{z} - 1 + z - z^2 + \cdots .
\]

\[\Rightarrow \text{Res }_{z=0} \frac{1}{z(z+1)} = 1.\]

\( z_0 = -1:\)

\[
\frac{1}{z(z+1)} = \frac{1}{z+1} \cdot \frac{1}{z+1-1} \\
= -\frac{1}{z+1} \cdot \frac{1}{1-(z+1)} \\
= -\frac{1}{z+1} \left(1 - (z+1) + (z+1)^2 - (z+1)^3 + \cdots \right) \\
= -\frac{1}{z+1} + 1 - (z+1) + \cdots .
\]

\[\Rightarrow \text{Res }_{z=-1} \frac{1}{z(z+1)} = -1.\]

So, \[\int_C \frac{1}{z(z+1)} \, dz = 2\pi i(1 + -1) = [0].\]

(2) Find \[\int_C \frac{z-1}{z(z+2)^2} \, dz, \] where \( C \) is the circle \(|z| = 4\).

\( f(z) = \frac{z-1}{z(z+2)^2} \) has two singularities, \( z_0 = 0 \) and \( z_0 = -2 \), both of which are inside \( C \).

\( z_0 = 0: \) First, \[\frac{z-1}{z(z+2)^2} = \frac{1}{(z+2)^2} - \frac{1}{z} \cdot \frac{1}{(z+2)^2}. \] Since

\[
\frac{1}{z+2} = \frac{1}{2} \left(1 + \frac{z}{2}\right) \\
= \frac{1}{2} \cdot \frac{1}{1 - \left(-\frac{z}{2}\right)} \\
= \frac{1}{2} \left(1 - \frac{z}{2} + \left(\frac{z}{2}\right)^2 - \cdots \right).
\]
So,

\[
\frac{1}{(z+2)^2} = \left[ \frac{1}{2} \left( 1 - \frac{z}{2} + \left( \frac{z}{2} \right)^2 - \cdots \right) \right]^2
= \frac{1}{4} \left( 1 - \frac{z}{2} + \left( \frac{z}{2} \right)^2 - \cdots \right) \left( 1 - \frac{z}{2} + \left( \frac{z}{2} \right)^2 - \cdots \right)
= \frac{1}{4} \left( 1 - z + \frac{3}{4}z^2 + \cdots \right)
= \frac{1}{4} - \frac{1}{4}z + \frac{3}{4}z^2 + \cdots .
\]

We thus see that

\[
\frac{1}{z} \cdot \frac{1}{(z+2)^2} = \frac{1}{4} \cdot \frac{1}{z} - \frac{1}{4} + \frac{3}{4}z - \cdots .
\]

Therefore,

\[
\text{Res}_{z=0} \frac{z-1}{z(z+2)^2} = 0 - \frac{1}{4} = -\frac{1}{4}.
\]

\(z_0 = -2\): In this case, we want

\[
\frac{z-1}{z(z+2)^2} = \frac{1}{(z+2)^2} \cdot \frac{z-1}{z}
= \frac{1}{(z+2)^2} \left( 1 - \frac{1}{z} \right).
\]

Since

\[
\frac{1}{z} = \frac{1}{(z+2) - 2}
= -\frac{1}{2} \cdot \frac{1}{1 - \left( \frac{z+2}{2} \right)}
= -\frac{1}{2} \left( 1 + \frac{z+2}{2} + \left( \frac{z+2}{2} \right)^2 + \left( \frac{z+2}{2} \right)^2 + \cdots \right)
= -\frac{1}{2} - \frac{1}{4}(z+2) - \frac{1}{8}(z+2)^2 - \frac{1}{16}(z+2)^3 - \cdots ,
\]

we obtain

\[
\frac{z-1}{z(z+2)^2} = \frac{1}{(z+2)^2} \left( 1 - \left( -\frac{1}{2} - \frac{1}{4}(z+2) - \frac{1}{8}(z+2)^2 - \frac{1}{16}(z+2)^3 - \cdots \right) \right)
= \frac{1}{(z+2)^2} \left( \frac{3}{2} + \frac{1}{4}(z+2) + \frac{1}{8}(z+2)^2 + \frac{1}{16}(z+2)^3 + \cdots \right)
= \frac{3}{2(z+2)^2} + \frac{1}{4} \frac{1}{z+2} + \frac{1}{8} + \frac{1}{16}(z+2) + \cdots
\Rightarrow \text{Res}_{z=-2} \frac{z-1}{z(z+2)^2} = \frac{1}{4}.
\]
So,
\[ \int_{C} \frac{z - 1}{z(z + 2)^2} \, dz = 2\pi i \left( -\frac{1}{4} + \frac{1}{4} \right) = 0. \]

4 The Three Types of Isolated Singular Points – Section 72 of Brown and Churchill

Definitions.

1. If \( z_0 \) is an isolated singularity of \( f \) and if all but a finite number of the \( b_n \) in the Laurent series for \( f \) are zero, then \( z_0 \) is a pole of \( f \). If \( b_n = 0 \) for \( n \geq N \), then \( z_0 \) is a pole of order \( N \). If \( z_0 \) is a first order pole, then \( z_0 \) is called a simple pole.

2. If an infinite number of \( b_n \)'s are nonzero, then \( z_0 \) is an essential singularity.

3. If \( b_n = 0 \) for all \( n \), then \( z_0 \) is a removable singularity.

Idea: \( f \) has a pole of order \( k \) at \( z_0 \) if and only if its Laurent expansion about \( z_0 \) has the form

\[
\frac{b_k}{(z - z_0)^k} + \cdots + \frac{b_1}{z - z_0} + \sum_{n=0}^{\infty} a_n (z - z_0)^n.
\]

Examples. Write the principal part of \( f \) and determine the type of singularity of the following.

(1) \( f(z) = \frac{\cos z}{z^2} \)

\( z_0 = 0 \) is the only singularity, and

\[
\begin{align*}
\frac{\cos z}{z^2} &= \frac{1}{z^2} \left( 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \right) \\
&= \frac{1}{z^2} - \frac{1}{2!} + \frac{z^2}{4!} - \cdots.
\end{align*}
\]

So, the principal part of \( f \) at \( z_0 = 0 \) is \( \frac{1}{z^2} \). Therefore, \( z_0 = 0 \) is a pole of order 2.

(2) \( f(z) = \frac{e^z - 1}{z^2} \)
\( z_0 = 0 \) is the only singularity, and
\[
\frac{e^z - 1}{z^2} = \frac{1}{z^2} \left( \left( 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \right) - 1 \right) \\
= \frac{1}{z^2} \left( z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \right) \\
= \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \cdots .
\]

So, the principal part of \( f \) at \( z_0 = 0 \) is \( \frac{1}{z} \). Therefore, \( z_0 = 0 \) is a simple pole.

(3) \( f(z) = \frac{\sin z}{z} \)

\( z_0 = 0 \) is the only singularity, and
\[
\frac{\sin z}{z} = \frac{1}{z} \left( z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \right) \\
= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots .
\]

In this case, the principal part of \( f \) at \( z_0 = 0 \) is 0, and \( z_0 = 0 \) is a removable singularity.

(4) \( f(z) = e^{i\frac{z}{z}} \)

Again, \( z_0 = 0 \) is the only singularity, and
\[
e^{i\frac{z}{z}} = 1 + \frac{1}{z} + \frac{1}{2!} \cdot \frac{1}{z^2} + \cdots .
\]

So, the principal part of \( f \) at \( z_0 = 0 \) is
\[
\frac{1}{z} + \frac{1}{2!} \cdot \frac{1}{z^2} + \cdots ,
\]

which has infinitely many terms, so \( z_0 = 0 \) is an essential singularity.

**NOTE:** If \( z_0 \) is a removable singularity of \( f \), then
\[
f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n,
\]
a convergent power series. If we define \( f(z_0) = a_0 \), then \( f \) is analytic at \( z_0 \).

5  Residues at Poles – Sections 73-74 of Brown and Churchill

A convenient way to determine residues at poles is described in the following theorem.
Theorem 2. An isolated singular point $z_0$ of a function $f$ is a pole of order $m$ if and only if $f(z)$ can be written in the form

$$f(z) = \frac{\phi(z)}{(z - z_0)^m},$$

where $\phi(z)$ is analytic at $z_0$ and $\phi(z_0) \neq 0$. Moreover,

- $\text{Res}_{z=z_0} f(z) = \phi(z_0)$ if $m = 1$, and
- $\text{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$ if $m \geq 2$.

Examples. Find the residues of the following functions.

1. $f(z) = \frac{z}{z^2 + 1}$

   The singularities of $f(z)$ are $z_0 = \pm i$, both of which are simple poles.

   $z_0 = -i$:
   $$f(z) = \frac{z}{(z + i)(z - i)} = \frac{z}{z - i} = \frac{\phi(z)}{z - i},$$

   Then, $\phi(z)$ is analytic at $z_0 = -i$ and $\phi(-i) \neq 0$.

   $$\Rightarrow \text{Res}_{z=-i} f(z) = \phi(-i) = \frac{-i}{-i - i} = \frac{1}{2}.$$ 

   $z_0 = i$:
   $$f(z) = \frac{z}{(z + i)(z - i)} = \frac{z}{z + i} = \frac{\phi(z)}{z + i},$$

   Then, $\phi(z)$ is analytic at $z_0 = i$ and $\phi(i) \neq 0$.

   $$\Rightarrow \text{Res}_{z=i} f(z) = \phi(i) = \frac{i}{i + i} = \frac{1}{2}.$$

2. $f(z) = \frac{z^2}{(z - 1)^3(z + 1)}$

   $f(z)$ has two singularities, $z_0 = -1$ and $z_0 = 1$.

   $z_0 = -1$:
   $$f(z) = \frac{\frac{z^2}{z + 1}}{(z - 1)^3} = \frac{\phi(z)}{z + 1} \Leftrightarrow z_0 = -1 \text{ is a simple pole.}$$

   Since $\phi(z)$ is analytic at $z_0 = -1$ and $\phi(-1) \neq 0$,

   $$\text{Res}_{z=-1} f(z) = \phi(-1) = \frac{(-1)^2}{(-1 - 1)^3} = -\frac{1}{8}.$$
\[ z_0 = 1: \]
\[ f(z) = \frac{z^2}{(z+1)(z-1)^3} = \frac{\phi(z)}{(z-1)^3} \]
\[ \iff z_0 = 1 \text{ is a pole of order } 3. \]

Since \( \phi(z) \) is analytic at \( z_0 = 1 \) and \( \phi(1) \neq 0, \)
\[ \text{Res}_{z=1} f(z) = \frac{\phi''(z)}{2!} \bigg|_{z=1}. \]

We have
\[ \phi(z) = \frac{z^2}{z+1} \Rightarrow \phi'(z) = \frac{2z + z^2}{(z+1)^2} \Rightarrow \phi''(z) = \frac{2}{(z+1)^3}. \]
\[ \implies \text{Res}_{z=1} f(z) = \frac{1}{2} \cdot \frac{2}{(1+1)^3} = \frac{1}{8}. \]

6 Zeros of Analytic Functions – Section 75 of Brown and Churchill

Suppose that \( f \) is analytic at a point \( z_0 \). If \( f(z_0) = 0 \) and \( f^{(j)}(z_0) = 0 \) for \( j = 1, 2, \ldots, m - 1 \), and \( f^{(m)}(z_0) \neq 0 \), then \( z_0 \) is a zero of order \( m \).

**Theorem** 3. Let \( f \) be a function analytic at a point \( z_0 \). \( f \) has a zero of order \( m \) at \( z_0 \) if and only if there exists a function \( g \) analytic at \( z_0 \) with \( g(z_0) \neq 0 \) such that
\[ f(z) = (z - z_0)^m g(z). \]

**Example.** Consider the function \( f(z) = z^3 - 4z^2 + 4z. \)

We can write \( f(z) = z(z-2)^2. \)

Therefore, has a zero of order \( m = 2 \) at \( z_0 = 2 \) and a zero of order \( m = 1 \) at \( z_0 = 0. \)

**Note:** If \( f \) is an analytic function not identically 0, then its zeros are isolated.

7 Zeros and Poles – Section 76 of Brown and Churchill

The following theorem gives us a way to determine poles for quotients of analytic functions.

**Theorem** 4. Suppose two functions \( p \) and \( q \) are analytic at a point \( z_0 \), \( p(z_0) \neq 0 \), and \( q \) has a zero of order \( m \) at \( z_0 \). Then, \( f(z) = \frac{p(z)}{q(z)} \) has a pole of order \( m \) at \( z_0. \)
Proof. Since \( q(z) \) has a zero of order \( m \) at \( z_0 \), we can write \( q(z) = (z-z_0)^m g(z) \), where \( g(z_0) \neq 0 \). Then,

\[
f(z) = \frac{p(z)}{q(z)} = \frac{p(z)}{(z-z_0)^m g(z)} = \frac{p(z)}{(z-z_0)^m} g(z) = \frac{\phi(z)}{(z-z_0)^m},
\]
where \( \phi(z) \) is analytic and nonzero at \( z_0 \). Therefore, \( z_0 \) is a pole of order \( m \). \( \square \)

Example. Determine the order of the pole \( z_0 = 0 \) for

\[
f(z) = \frac{z - 1}{\sin^2 z}.
\]

Define \( p(z) = z - 1 \) and \( q(z) = \sin^2 z \). Then,

- \( p \) and \( q \) are entire (and so analytic at \( z_0 = 0 \)).
- What order zero of \( q(z) \) is \( z_0 = 0 \)?
  \[
  q(z) = \sin^2 z \rightarrow q(0) = 0 \\
  q'(z) = 2 \sin z \cos z = \sin 2z \quad \Longrightarrow \quad q'(0) = 0 \\
  q''(z) = 2 \cos 2z \quad \Longrightarrow \quad q''(0) = 2 \neq 0.
  \]
So, \( z_0 = 0 \) is a zero of order 2 \( \Longrightarrow \) \( z_0 = 0 \) is a pole of \( f \) of order 2.

Theorem 5. Let functions \( p \) and \( q \) be analytic at a point \( z_0 \). If \( p(z_0) \neq 0 \), \( q(z_0) = 0 \), and \( q'(z_0) \neq 0 \), then \( z_0 \) is a simple pole of \( \frac{p(z)}{q(z)} \), and

\[
\text{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}.
\]

Examples. Find the residues of the following functions at the indicated point.

(1) \( \frac{e^{z^2}}{z - 1}, \ z_0 = 1 \)

\[
\frac{e^{z^2}}{z - 1} = \frac{p(z)}{q(z)} \\
p(1) = e^{1^2} = e^1 \neq 0 \quad \text{and} \quad p, \ q \ \text{are entire.} \\
q(1) = 0 \\
q'(z) = 1 \quad \Longrightarrow \quad q'(1) = 1 \neq 0.
\]
So,

\[
\text{Res}_{z=1} \frac{e^{z^2}}{z - 1} = \frac{p(1)}{q'(1)} = \frac{e^1}{1} = e^1.
\]
(2) \( \frac{z + 2}{z^2 - 2z}, \ z_0 = 0 \)

\[
\frac{z + 2}{z^2 - 2z} = \frac{p(z)}{q(z)}
\]

\( \implies \) \( p, q \) are entire.

\[
p(0) = 2 \neq 0 \\
q(0) = 0 \\
q'(z) = 2z - 2 \implies q'(0) = -2 \neq 0.
\]

So,

\[
\text{Res}_{z=0} \frac{z + 2}{z^2 - 2z} = \frac{p(0)}{q'(0)} = \frac{2}{-2} = -1.
\]

(3) \( \frac{z^2}{z^4 - 1}, \ z_0 = i \)

\[
\frac{z^2}{z^4 - 1} = \frac{p(z)}{q(z)}
\]

\( \implies \) \( p, q \) are entire.

\[
p(i) = i^2 = -1 \\
q(i) = 0 \\
q'(z) = 4z^3 \implies q'(i) = 4(i^3) = -4i \neq 0.
\]

So,

\[
\text{Res}_{z=i} \frac{z^2}{z^4 - 1} = \frac{p(i)}{q'(i)} = \frac{-1}{-4i} = \frac{-i}{4}.
\]